M 332 Unit 14

THE OPEN UNIVERSITY
Complex Analysis
Mathematics : A Third Level Course

Unit 14 Laplace Transforms

COMPLEX ANALYSIS

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

Complex Analysis Unit 14

Laplace Transforms

Prepared by the Course Team

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1.1

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Conventions

Before working through this text make sure you have read *A Guide to the Course: Complex Analysis*.

References to units of other Open University courses in mathematics take the form:

Unit M100 13, Integration II.

The set book for the course M231, Analysis, is M. Spivak, *Calculus*, paperback edition (W. A. Benjamin/Addison-Wesley, 1973). This is referred to as:

Spivak.

Optional Material

This course has been designed so that it is possible to make minor changes to the content in the light of experience. You should therefore consult the supplementary material to discover which sections of this text are not part of the course in the current academic year.

Reminder: This unit is one of the last four units in the course. You are not required to study all of these four units, so you should consult the supplementary material to see what choice is available to you.

14.0 INTRODUCTION

This unit is concerned with Laplace Transforms and Asymptotic Expansions; these topics provide two important tools of the applied mathematician. The use of these tools, and the justification for their use, require a considerable knowledge of complex analysis and an ability to apply that knowledge. In one unit, it is not possible to give a systematic introduction to either of these topics, let alone both. But that is not the object of the exercise. Our purpose, as with the other three units in this last group of four, is to show you something of complex analysis at work. To serve this end, we have tried to give just enough background of Laplace Transforms and of Asymptotic Expansions to reveal the type of analytic problem that arises and the usefulness of some of the theorems and techniques that we have developed in the course.

This unit may appear to you to be rather long. But it is not so much a "doing" unit as others in the course and there are fewer problems for you to tackle. Some sections are not followed by a problems section, not because there are no interesting problems available, but because the material is essentially background material and we do not want to overburden you with *too* much work!

In some of the proofs in this unit we have to call on some ideas and results of real analysis which may be outside your experience. They are not essential to your understanding of the essence of the unit and if you have not met them you will not be disadvantaged in tackling problems on the unit. For convenience we have listed these results in an appendix—see page 71.

The main work in the unit is presented in Sections 14.2, 14.4, 14.7 and 14.9. In Section 14.2 the validity of the inversion integral for Laplace transforms is discussed and in Section 14.4 some methods for evaluating the integral in special cases are given. In Section 14.7, the idea of an asymptotic expansion is introduced, together with Watson's Lemma, which gives a method of finding asymptotic expansions of certain integrals having the form of Laplace transforms. Another special technique for finding asymptotic expansions, the Method of Steepest Descent, is given in Section 14.9.

In order to gain some idea of what is to come *glance over* Example 8 (page 20), which illustrates the utility of the Laplace transform to solve a differential equation, and pages 44 and 45 (as far as the rule) where the idea of an asymptotic expansion is introduced.

The problems sections indicate some of the sort of things we expect you to be able to do in the examination and assessment TMA's. The following list of objectives indicates the performance you should aim for.

Objectives

After working through the unit, you should be able to:

(i) recall the definition of the Laplace transform of a function, outline its use in the solution of differential equations, and apply it in simple cases;

(ii) for given functions F determine whether the inversion integral

$\frac{1}{2\pi i}\lim_{r\to\infty}\int_{\gamma-ir}^{\gamma+ir} e^{pt}F(p)\,dp$ exists (by checking the conditions of Theorem 1).

(iii) given the function F, evaluate the inversion integral

$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,F(p)\,dp$ by using the Residue Theorem, and applying Theorem 6 in appropriate cases;

(iv) recall the definition of asymptotic expansion, compare it with the definition of convergent series, and apply it in simple cases;

(v) explain and use Watson's Lemma in determining the asymptotic expansion of a Laplace transform;

(vi) use the method of steepest descent to calculate asymptotic expansions to integrals of the form $\int_\Gamma e^{ph(z)} f(z)\,dz$;

(vii) apply the appropriate formula for the dominant term in an asymptotic expansion.

Results Needed from Earlier Units

Unit 3: Section 3.2—analytic functions preserve angles.
Unit 4: The Estimation Theorem.
Unit 5: Cauchy's Formula.
Unit 6: (i) Taylor's Theorem.
(ii) The Uniqueness Theorem.
Unit 9: (i) Theorem 1.
(ii) The Maximum Principle.
(iii) Problem 3 of Section 9.2.
(iv) Cauchy's Theorem.
(v) The Residue Theorem.
Unit 10: (i) Theorem 2.
(ii) Section 10.1—residues for poles of order >2.
(iii) Section 10.4—integration techniques.
(iv) Problem 1 of Section 10.5.
(v) Problem 4 of Section 10.5.
Unit 11: (i) Example 2 of Section 11.3.
(ii) Section 11.8—the Gamma Function, and the Schwarz Reflection Principle.

Radio

There is a radio programme associated with this unit.

14.1 INTEGRAL TRANSFORMS

In this first section, we shall discuss the idea of an integral transform in a general way, without going into any rigorous mathematical detail. You may well have met some of these ideas before, especially if you have studied M201, Linear Mathematics. However, you need not consider it a disadvantage if the material is new to you; the fact that the discussion in this section will be rather cursory is not to be taken as an indication that we expect you to know anything about the subject before you start. The object of this section is to give you a broad picture in which you can place the more detailed work that will be presented in the following section.

* * * * * * * *

An **integral transform** of a function f is defined to be a function F specified by a formula of the form

$$F(p) = \int_a^b K(p, x) f(x)\, dx,$$

where K, a function of two variables, is called the **kernel** of the transform. The purpose of introducing a transform is simply that, by a suitable choice of the kernel and of a and b, the function F often turns out to be simpler to deal with than the function f. (We shall see some specific examples of this later.) This idea is certainly *not* new to you; K, a and b define a mapping which takes f to F and, hopefully, this mapping will be a morphism, for then the relevant operation defined on the functions in the domain has a counterpart in the set of transformed functions, and this enables the problem to be transferred completely to this new set. So the idea is no more difficult than that which underlies the use of logarithms; but putting the idea into practice can, of course, be much more complicated, simply because we shall be dealing with problems which are harder than just multiplying two numbers together.

The use of integral transforms can occur in two ways: (i) in providing a simpler method of solution for problems that are tractable by some other means; (ii) in providing a method for the solution of problems which are otherwise intractable.

There is a similarity between integral transforms and logarithms. Once the problem has been solved in terms of the transformed function, the solution has to be interpreted in terms of the original function; in other words, we need a method of recovering the original function from a knowledge of its transform. In logarithms, the recovery is effected by the inverse function, the exponential. For integral transforms we shall have a similar problem and we shall have to find an *inversion formula.* The following example provides an example of the idea of an inversion formula in a slightly different context.

Example 1

You may recall (and it doesn't matter if you don't) that a differential equation of the form

$$y'(x) + p(x)y(x) = q(x)$$

can be solved (that is, an explicit formula for $y(x)$ can be found) by using an *integrating factor* of the form $e^{\int p(x)dx}$. Multiplication by this factor gives

$$y'(x)e^{\int p(x)\,dx} + p(x)e^{\int p(x)\,dx}y(x) = q(x)e^{\int p(x)dx},$$

which is equivalent to

$$\left(y(x)e^{\int p(x)\,dx}\right)' = q(x)e^{\int p(x)dx}.$$

In other words,

$$y(x)e^{\int p(x)dx} = \int q(x)e^{\int p(x)dx}\, dx.$$

In this example we are not using an *integral* transform to solve a differential equation, but we are using a kind of transform—we are solving for the function z where $z(x) = y(x)e^{\int p(x)\,dx}$ because the problem in z is easier than the problem in y. The "inversion formula" is, of course, just

$$y(x) = e^{-\int p(x)\,dx} z(x),$$

which gives the solution we are looking for.

Finite Fourier Transforms

An example of a *genuine* integral transform arises from an idea that you may have met elsewhere (M201, Linear Mathematics, MST282, Mechanics and Applied Calculus, M321, Partial Differential Equations of Applied Mathematics, SM351 Quantum Theory and Atomic Structure, for example). The idea is that of expressing a function by a *Fourier series**. If you have not met the idea before, it does not really matter provided that you are willing to accept the following statement.

If a function f can be represented by a trigonometric series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \tag{1}$$

then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \tag{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n \in \mathbf{N}, \tag{3}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n \in \mathbf{N}. \tag{4}$$

The formulas for a_n and b_n have the appearance of integral transforms; comparing (3) and (4) with the general integral transform

$$\int_a^b K(p, x) f(x)\,dx,$$

we have $a = -\pi$, $b = \pi$ and $K(p, x)$ is respectively $\dfrac{\cos nx}{\pi}$ and $\dfrac{\sin nx}{\pi}$. ($n = 0$ is a special case.) The symbol n is used to emphasize that p is taking integer values—a condition certainly not required of an integral transform in general.

Important special cases arise when f is either an odd function or an even function.

(i) If f is even, then

$$b_n = 0, \quad n \in \mathbf{N},$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx,$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad n \in \mathbf{N}$$

(since, if g is an even function, then $\displaystyle\int_{-a}^{a} g(x)\,dx = 2\int_0^a g(x)\,dx$).

* *Joseph Fourier* (1768–1830) was one of the outstanding mathematicians in the early years of the Ecole Polytechnique (in Paris). He worked in many fields, notably heat conduction. He used "Fourier series" to solve the associated partial differential equations. His manipulation of Fourier series caused mathematicians to question just what is meant by "a function".

(ii) If f is an odd function, then

$$a_n = 0, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \quad n \in \mathbf{N}.$$

If we know f then we can use the formulas (2), (3) and (4) to construct the sequences $\{a_n\}$ and $\{b_n\}$. On the other hand, as we shall see later, it is often easier to construct the sequences $\{a_n\}$ and $\{b_n\}$ and use the formulas as a recipe for finding f. In terms of integral transforms we would then be regarding the formulas for a_n and b_n as integral transforms, and the series (1) as a formula that reinstates the function f from a_n and b_n and so gives us an inversion formula.

For example, if f is known to be an odd function, we can use the *sine transform*, F_s, given by

$$F_s(n) = \int_0^{\pi} f(x)\sin nx\,dx, \quad n \in \mathbf{N},$$

so that $\frac{2}{\pi}F_s(n)$ is b_n. The inversion formula is, from (1),

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n)\sin nx. \tag{5}$$

If f is an even function, we can use the *cosine transform*, F_c, given by

$$F_c(n) = \int_0^{\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \ldots;$$

the inversion formula is, from (1),

$$f(x) = \frac{1}{\pi}F_c(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} F_c(n)\cos nx. \tag{6}$$

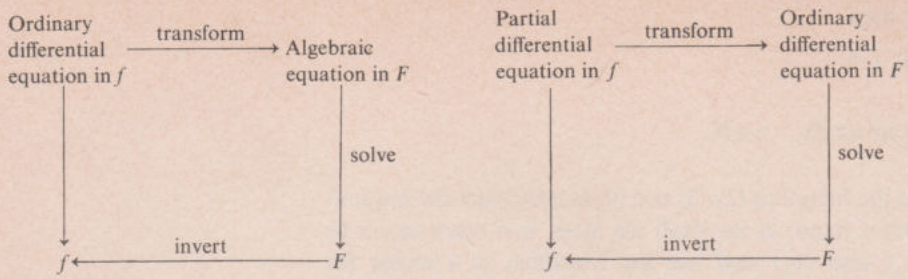
We shall establish some results we shall need in Example 3. We often need to find the transforms of the derivatives of f, in particular f''. Straightforward integration by parts gives the sine and cosine transforms of f'', respectively, as

$$\int_0^{\pi} f''(x)\sin nx\,dx = -n^2F_s(n) + n[f(0) - (-1)^n f(\pi)], \tag{7}$$

$$\int_0^{\pi} f''(x)\cos nx\,dx = -n^2F_c(n) + (-1)^n f'(\pi) - f'(0). \tag{8}$$

Notice that in (7) values of the *function* at 0 and π are involved, whereas in (8) values of the *derivative* at 0 and π occur. It is fairly clear then that the sine transform may well be applicable in cases where the values of the function at 0 and π are known, whereas the cosine transform is likely to come into its own in cases where the values of the derivatives are known at 0 and π.

In view of the results for the transforms of f'' it is possible to "take the transform" of a differential equation and in so doing replace the operation of double differentiation by the *algebraic* operation of multiplication of F_s (or F_c) by $-n^2$. For an *ordinary* differential equation, the consequence is that one is presented with the problem of solving an *algebraic* equation for the transformed function. For a *partial* differential equation, it turns out that one has to solve an *ordinary* differential equation for the transformed function. In either case the prospects are that the problem in F_s or F_c is an easier one than that in f itself. The function f is reinstated from F_s or F_c by using the inversion formula. The two cases are summed up in the following schema.



It is possible to extend the sine and cosine transforms in a straightforward way to intervals other than $[0, \pi]$ and in a rather more subtle way to sets of values of n other than the integers. This latter generalization makes possible the application of the method to cases where a mixture of the function values themselves and the values of the derivatives are known at the end points of the interval.

The derivations of the inversion formulas (5) and (6) depend upon the results

$$\int_0^\pi \sin mx \sin nx \, dx = \int_0^\pi \cos mx \cos nx \, dx = 0, \quad m \neq n,$$

which are crucial in the calculation of the coefficients a_n and b_n. This property is referred to as **orthogonality**: the sets of functions $\{x \longrightarrow \sin mx\}$ and $\{x \longrightarrow \cos nx\}$ are each said to be **orthogonal** in $[0, \pi]$.

Infinite Transforms

So far we have discussed transforms for which the limits of integration are finite; such transforms are called *finite integral transforms*. They are useful but do not really make tractable any problems that cannot be solved by direct methods.* It *is* possible to solve some new problems, however, by using *infinite transforms*—integral transforms where one or both of the limits of integration becomes infinite. As an example of such a transform, we shall consider a generalization of the (finite) sine and cosine transforms.

The success of the (finite) sine and cosine transforms depends upon the fact that a large class of functions can be represented as series of sines or as series of cosines. But the trigonometric functions are periodic with period 2π and so the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \sin nx$$

can be used to represent only functions which are also periodic or defined on a finite interval of length 2π.

It is fairly straightforward to show that, given l, the sets $\{x \longrightarrow \sin n\pi x/l\}$ and $\{x \longrightarrow \cos n\pi x/l\}$ are also orthogonal and these enable us to represent suitable functions with period $2l$ or defined on an interval of length $2l$. If

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l},$$

then

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx,$$

* Since the solution turns out to be of series form, one can employ straightforward methods to determine the coefficients. The effectiveness of the integral transforms lies in being able to adopt a more formal approach to the solution and so their value tends to increase with the complexity of the problem.

and if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

then

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Other functions can be represented as series of sines and cosines together, in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

in some interval $(-l, l)$. We can write this in a more concise form as

$$f(x) = \frac{a_0}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \exp\left(\frac{in\pi x}{l}\right), \tag{9}$$

where, with n positive, $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$, and so

$$c_n = \tfrac{1}{2}(a_n - ib_n) \text{ and } c_{-n} = \tfrac{1}{2}(a_n + ib_n), \quad n = 1, 2, 3, \ldots.$$

The formulas for a_n and b_n combine to give

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\exp\left(-\frac{in\pi x}{l}\right)dx, \quad n = 1, 2, 3, \ldots, \tag{10}$$

and $c_0 = \dfrac{a_0}{2}$.

As we have remarked, the trigonometric series can be used to represent only functions that are periodic or defined on a finite interval. This is a drawback, but one that can be overcome, at least formally, as follows.

If we write the formulas for a_0 and c_n in the series expansion (9) for $f(x)$, we get

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2l}\int_{-l}^{l} f(t)\exp\left(-\frac{in\pi(t-x)}{l}\right)dt,$$

where we have used t as the variable of integration when substituting (10) into (9). We can change the appearance of this integral, by writing

$$p_n = \frac{n\pi}{l} \quad \text{and} \quad g(p_n) = \int_{-l}^{l} f(t)\exp\big(ip_n(x-t)\big)\,dt;$$

the infinite sum then has the form

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(p_{n+1} - p_n)g(p_n).$$

This is a form similar to that which occurs in the definition of the Riemann integral and suggests that the limit of this expression as l increases is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(p)\,dp.$$

Furthermore, $g(p)$ would become

$$\int_{-\infty}^{\infty} f(t)\exp\big(ip(x-t)\big)\,dt;$$

also, if $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt$ exists, then

$$\lim_{l\to\infty}\frac{1}{2l}\int_{-l}^{l} f(t)\,dt = 0.$$

Putting all this together would give

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\exp(ip(x-t))\,dt\right)dp. \tag{11}$$

(The order of operations in this *repeated integral* is: perform the integral with respect to t (in the large parentheses) and then integrate the result with respect to p.)

Of course, there is a great deal of work to do in order to justify these manipulations. In fact, a rigorous justification is very difficult and a better proposition is to attempt to perform a *direct* analysis of the final result and at least justify the formula, which is after all the object of the manipulations. The final formula (11) is the *exponential form of Fourier's Integral Formula* and from our point of view its importance lies in the fact that it presents us with the possibility of defining a transform over the interval $(-\infty, \infty)$. If we write

$$F(p) = \int_{-\infty}^{\infty} f(t)\exp(-ipt)\,dt, \tag{12}$$

where F is called the **Fourier transform** of f, then the associated **inversion formula** may be obtained directly from Fourier's Integral Formula (11): it is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\exp(ipx)\,dp. \tag{13}$$

The factor $1/(2\pi)$ can be distributed according to taste; it can be used as here, in the inversion formula (13), or taken with the integral transform itself (12). There is a certain aesthetic charm in taking $1/(\sqrt{2\pi})$ with each integral. The form given here, however, is particularly useful in some problems (for example, those involving frequency analysis), where p is usually replaced by $2\pi\omega$ and then the factor $1/(2\pi)$ in the inversion formula is absorbed in the change of variable. But *our* reasons are rather less subtle; this particular form—(12) and (13)—is useful in revealing the connection between the Fourier transform and the Laplace transform.

Example 2

In this example we shall find the Fourier transform of a function f and then invert it, thus recovering f.

Let $f(x) = e^{-a|x|}$, $x \in \mathbf{R}$, where $a > 0$. Then, from (12), we have

$$\begin{aligned} F(p) &= \int_{-\infty}^{\infty} e^{-a|t|}\,e^{-ipt}\,dt \\ &= \int_{-\infty}^{0} e^{(a-ip)t}\,dt + \int_{0}^{\infty} e^{-(a+ip)t}\,dt \\ &= \frac{2a}{a^2+p^2}. \end{aligned}$$

Thus $p \longrightarrow 2a/(a^2 + p^2)$ is the Fourier transform of f. To check the inversion formula, we can substitute the expression for $F(p)$ and see whether $f(x)$ is reinstated. From (13), we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\exp(ipx)\,dp = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}}{a^2+p^2}\,dp.$$

The integrand has simple poles at $\pm ia$ and we can evaluate the integral using the techniques of Section 10.4 of *Unit 10, The Calculus of Residues*, by choosing a closed semicircular contour with diameter the interval $[-r, r]$ of the real axis. The curved part of the contour will have to be chosen so that the integral along that part approaches zero as r increases. The sign of x is critical. If $x < 0$, we choose a contour in the lower half-plane. If $x > 0$, we choose a contour in the upper half-plane. See Fig. 1.

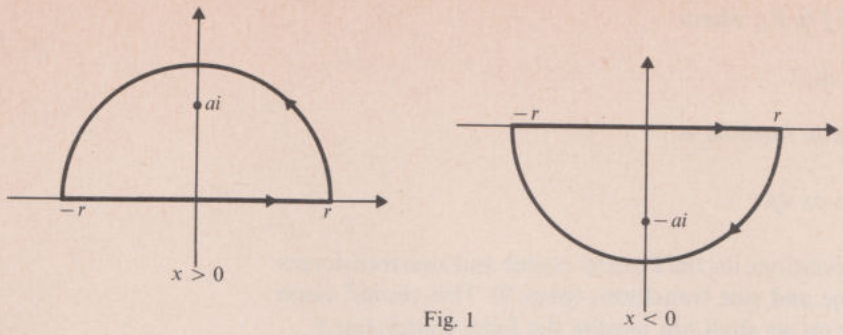


Fig. 1

The usual types of estimation shows that in each case the integral along the curved part of the contour approaches zero and so we get

$$\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{ipx}}{a^2+p^2}\,dp = \begin{cases} -\dfrac{a}{\pi}\cdot 2\pi i \operatorname{Res}(p \longrightarrow e^{ipx}/(a^2+p^2), -ai), & x<0, \\ \dfrac{a}{\pi}\cdot 2\pi i \operatorname{Res}(p \longrightarrow e^{ipx}/(a^2+p^2), ai), & x>0. \end{cases}$$

(Note the minus sign for $x < 0$, in view of the fact that the contour is traversed in the negative direction.)

From Theorem 2 of *Unit 10* the residue at $-ai$ is $-e^{ax}/(2ia)$ and the residue at ai is $e^{-ax}/(2ia)$ and so the value of the integral is e^{ax} if x is negative and e^{-ax} if x is positive, that is to say, it is $e^{-a|x|}$ which, much to our relief, is the function we first thought of. (We have left the case $x = 0$ for you to verify.)

We get a useful alternative form of Fourier's Integral Formula (11) by putting the exponential term back into trigonometric form and then using the fact that sine and cosine are respectively odd and even functions. The form so obtained is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\big(p(x-t)\big)\,dt\right)dp.$$

Two other forms can be obtained in the cases where the function f is either an even function or an odd function. On expanding the cosine term, we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(t)(\cos px\cos pt + \sin px\sin pt)\,dt\right)dp;$$

so, if f is even,

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos px\left(\int_0^{\infty} f(t)\cos pt\,dt\right)dp,$$

and if f is odd,

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin px\left(\int_0^{\infty} f(t)\sin pt\,dt\right)dp.$$

These two formulas are called respectively *Fourier's cosine formula* and *Fourier's sine formula* and they give us the extension of the finite cosine and sine transforms (page 9) that we were looking for. If we write

$$F_c(p) = \int_0^{\infty} f(t)\cos pt\,dt,$$

then F_c is the **Fourier cosine transform** of f and

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(p)\cos px\,dp$$

is the corresponding inversion formula.

The **Fourier sine transform** of f is F_s, where

$$F_s(p) = \int_0^\infty f(t) \sin pt \, dt,$$

and the corresponding inversion formula is

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(p) \sin px \, dp.$$

(We have used the same abbreviations for the Fourier cosine and sine transforms as we used for the finite cosine and sine transforms (page 9). This should cause no confusion since from here on we shall not require the finite transforms.)

The following example concerns a partial differential equation. Recall that we define the *partial derivatives* of the real-valued function F *at the point* (a, b), denoted by $\frac{\partial F}{\partial x}(a, b)$ and $\frac{\partial F}{\partial y}(a, b)$, to be the real numbers

$$\frac{\partial F}{\partial x}(a, b) = \lim_{h \to 0} \frac{F(a + h, b) - F(a, b)}{h}$$

and

$$\frac{\partial F}{\partial y}(a, b) = \lim_{k \to 0} \frac{F(a, b + k) - F(a, b)}{k} \quad \text{(where } h, k \in \mathbf{R}\text{)}.$$

The *functions* $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are called the *partial derivatives* of F. These functions are real-valued, and so may well have partial derivatives; for example, the partial derivatives of $\frac{\partial F}{\partial x}$ are $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$, and these are normally abbreviated $\frac{\partial^2 F}{\partial x^2}$ and $\frac{\partial^2 F}{\partial y \partial x}$ respectively.

The functions $\frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial x \partial y}, \frac{\partial^2 F}{\partial y \partial x}$ are called the *second partial derivatives* of F.

Example 3

The temperature in a semi-infinite plate $\{(x, y): y > 0\}$ is kept constant at 1 on the edge $y = 0$ from $x = -a$ to $x = a$ and zero on the edge $y = 0, |x| > a$. Find the steady-state temperature distribution. This is the temperature distribution to which the plate finally settles down. (This example is almost identical to Problem 2 in Section 16.8 of *Unit 16, Boundary Value Problems*.)

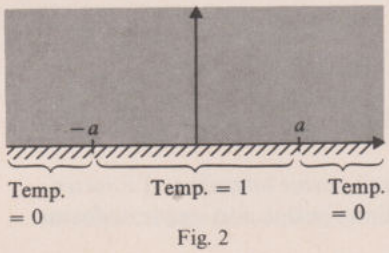


Fig. 2

The steady-state temperature function v will be independent of time and so will satisfy the partial differential equation

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0.$$

To solve this equation we will use the method outlined in the right-hand diagram on page 10, namely: transform the partial differential equation, solve the resulting ordinary differential equation and then invert this solution by means of an inversion formula to obtain the solution of the original partial differential equation.

Step 1 Transform the Partial Differential Equation

By symmetry, v will be an even function of x and so we try for a solution using a cosine transformation with respect to x; we obtain

$$\int_0^\infty \left(\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) \right) \cos px \, dx = 0. \tag{14}$$

Assuming that it is valid to take the derivative with respect to y "through the integral sign" (see item 5 of the list of real analysis results in the Appendix on page 71), we have for the second term of (14)

$$\int_0^\infty \frac{\partial^2 v}{\partial y^2}(x, y) \cos px \, dx = \frac{\partial^2}{\partial y^2} \int_0^\infty v(x, y) \cos px \, dx$$

$$= \frac{\partial^2 V}{\partial y^2}(p, y),$$

where $V(p, y) = \int_0^\infty v(x, y) \cos px \, dx$.

For the first term of (14), we have, on integrating by parts,

$$\int_0^\infty \frac{\partial^2 v}{\partial x^2}(x, y) \cos px \, dx = \cos px \frac{\partial v}{\partial x}(x, y) \Big|_0^\infty + p \int_0^\infty \frac{\partial v}{\partial x}(x, y) \sin px \, dx.$$

Now since v is an even function of x, $\frac{\partial v}{\partial x}(x, y) = 0$ at $(0, y)$. We may assume also that for all y, $\lim_{x \to \infty} \frac{\partial v}{\partial x}(x, y) = 0$ (corresponding to the physical condition that the temperature gradient in the x-direction is small far from the origin). Thus the first term on the right-hand side vanishes and we get, on integrating by parts,

$$\int_0^\infty \frac{\partial^2 v}{\partial x^2}(x, y) \cos px \, dx = pv(x, y) \sin px \Big|_0^\infty - p^2 \int_0^\infty v(x, y) \cos px \, dx$$

$$= -p^2 V(p, y),$$

since $v(x, y)$ will approach zero as x increases. The original equation thus transforms to

$$-p^2 V(p, y) + \frac{d^2 V}{dy^2}(p, y) = 0, \tag{15}$$

since for any fixed value of p, the derivative becomes an ordinary derivative.

Step 2 Solve the Ordinary Differential Equation

We deduce from (15) that

$$V(p, y) = Ae^{py} + Be^{-py},$$

where A and B are "constants" which may depend on p.

We now need some conditions on V that will enable us to determine A and B. From physical considerations we know that $\lim_{y \to \infty} v(x, y) = 0$ and we can show (although we shall not do so) that $\lim_{y \to \infty} V(p, y) = 0$. Thus, if we choose $p > 0$, we have $A = 0$, and so

$$V(p, y) = Be^{-py}.$$

We also know

$$v(x, y) = \begin{cases} 1, & |x| \leqslant a, y = 0 \\ 0, & |x| > a, y = 0. \end{cases}$$

So

$$V(p, 0) = \int_0^\infty \cos px\, v(x, 0)\, dx$$
$$= \int_0^a \cos px\, dx = \frac{\sin pa}{p}.$$

Thus $B = (\sin pa)/p$, and so

$$V(p, y) = \frac{\sin pa}{p} e^{-py}, \quad p > 0.$$

This solves our transformed problem.

Step 3 Use the Inversion Formula

We now use the inversion formula to give

$$v(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin pa}{p} e^{-py} \cos px\, dp$$
$$= \frac{1}{\pi} \int_0^\infty \frac{e^{-py}}{p} [\sin(p(a + x)) + \sin(p(a - x))]\, dp. \tag{16}$$

This can be split into two integrals, each of the form

$$\frac{1}{\pi} \int_0^\infty \frac{e^{-cp} \sin bp}{p} dp, \tag{17}$$

with $c > 0$.

(In the first integral $c = y$ and $b = a + x$, in the second $c = y$ and $b = a - x$. We replace y by c to emphasize that, as far as this integration is concerned, y is constant.)

We can evaluate this integral (17) using the techniques we developed in *Unit 10*. It can be written as

$$\frac{1}{\pi} \int_0^\infty \frac{\exp(-(c - ib)p) - \exp((-c - ib)p)}{2ip} dp.$$

Let $c + ib = Me^{i\alpha}$, $c - ib = Me^{-i\alpha}$; then the integral becomes

$$\frac{1}{2\pi i} \int_0^\infty [\exp(-Mpe^{-i\alpha}) - \exp(-Mpe^{i\alpha})] \frac{1}{p} dp$$

and with $Mp = r$ this becomes

$$\frac{1}{2\pi i} \int_0^\infty [\exp(-re^{-i\alpha}) - \exp(-re^{i\alpha})] \frac{1}{r} dr. \tag{18}$$

Now consider the integral $\int_\Gamma \frac{e^{-z}}{z} dz$, where Γ is the closed contour $ABDE$ shown in Fig. 3, which is the boundary of the annular sector of angle α and radii ε and ρ, where $\varepsilon < \rho$.

The integrand is analytic on the region inside Γ and so by Cauchy's Theorem, $\int_\Gamma \frac{e^{-z}}{z} dz = 0$; therefore

$$\int_{BD} \frac{e^{-z}}{z} dz + \int_{EA} \frac{e^{-z}}{z} dz = -\int_{AB} \frac{e^{-z}}{z} dz - \int_{DE} \frac{e^{-z}}{z} dz.$$

Now

$$\left| \int_{AB} \frac{e^{-z}}{z} dz \right| = \left| \int_{-\alpha}^{\alpha} e^{-\rho(\cos\theta + i \sin\theta)}\, d\theta \right| \leqslant e^{-\rho \cos\alpha} 2\alpha,$$

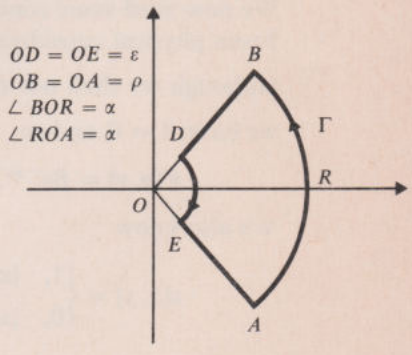


Fig. 3

and so this integral approaches zero as ρ increases provided $0 < \alpha < \pi/2$. (You may have noticed that we are implicitly assuming that $b > 0$. If you care to, you can check that with $b < 0$ you get the same answer.) The integral along DE can be evaluated using Problem 4 of Section 10.5 of *Unit 10*; its limiting value as ε approaches zero is $-2\alpha i \times$ the residue of $z \longrightarrow \dfrac{e^{-z}}{z}$ at 0 (which is just 1). Thus

$$\lim_{\varepsilon \to 0^+} \int_{DE} \frac{e^{-z}}{z}\, dz = -2\alpha i.$$

We can therefore conclude that

$$\lim_{\substack{\varepsilon \to 0^+ \\ \rho \to \alpha}} \left(\int_{BD} \frac{e^{-z}}{z}\, dz + \int_{EA} \frac{e^{-z}}{z}\, dz \right) = 2\alpha i.$$

But on BD, $z = re^{i\alpha}$ with r varying from ρ to ε, and on EA, $z = re^{-i\alpha}$ with r varying from ε to ρ. Thus

$$-\int_0^\infty \frac{\exp(-re^{i\alpha})}{re^{i\alpha}}\, e^{i\alpha}\, dr + \int_0^\infty \frac{\exp(-re^{-i\alpha})}{re^{-i\alpha}}\, e^{-i\alpha}\, dr = 2\alpha i,$$

and so

$$\frac{1}{2i}\int_0^\infty [\exp(-re^{-i\alpha}) - \exp(-re^{i\alpha})]\frac{1}{r}\, dr = \alpha.$$

Thus the integral (17), which equals the integral (18), has the value,

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-cp} \sin bp}{p}\, dp = \frac{\alpha}{\pi}.$$

But $\alpha = \text{Arg}(c + ib)$, and so we deduce that

$$\int_0^\infty \frac{e^{-cp} \sin bp}{p}\, dp = \text{Arg}(c + ib).$$

But $\tan \alpha = b/c$, because $c + ib = Me^{i\alpha}$, and so we deduce that

$$\int_0^\infty \frac{e^{-cp} \sin pb}{p}\, dp = \arctan \frac{b}{c}.$$

The integral in question (16) is therefore given by

$$\frac{1}{\pi}\int_0^\infty e^{-py}[\sin(p(a + x)) + \sin(p(a - x))]\frac{1}{p}\, dp$$

$$= \frac{1}{\pi}[\text{Arg}(y + i(a + x)) + \text{Arg}(y + i(a - x))]$$

and this is the temperature distribution $v(x, y)$.

Self-Assessment Question 1

Outline the main steps in the use of a Fourier transform to solve a partial differential equation.

Solution

(1) Transform partial differential equation to an ordinary differential equation by an appropriate Fourier transform.

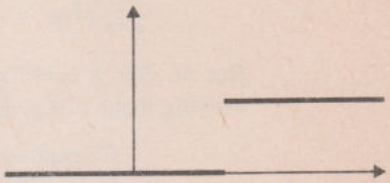
(2) Solve the ordinary differential equation.

(3) Use the appropriate inversion formula to obtain the solution of the partial differential equation from the solution of the ordinary differential equation.

The Laplace Transform

When we derived the Fourier integral formula (11), we passed over the analytic details. It turns out that the integral transforms are valid (in the sense that the integrals exist) and the inversion formula does reinstate the correct function if $\int_{-\infty}^{\infty} |f|$ exists and f is **piecewise continuous**, by which we mean that f is continuous at all but a finite number of points for *every* finite interval $[a, b]$; and if x_0 is a point of discontinuity then $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ both exist. At a point of discontinuity, the inversion formula will give the mean of the values of the left- and right-hand limits of the function at that point.

The condition that $\int_{-\infty}^{\infty} |f|$ should exist can be rather restrictive, especially in cases where the variable represents time. It is not satisfied, for example, by step functions (Fig. 4) which are associated with many kinds of physical events and discontinuities.



A step function

Fig. 4

To see how the Laplace transform* arises, we consider the function

$$g(x) = e^{-\gamma x} f(x),$$

where γ is real and positive. The Fourier inversion integral for g will exist if $\int_{-\infty}^{\infty} |e^{-\gamma x} f(x)|\, dx$ exists. If we assume further that $f(x) = 0$ for $x < 0$, as would be reasonable if x were representing time, then this latter integral would certainly exist if constants K and a could be found so as to make $|f(x)| < Ke^{ax}$, where $a < \gamma$.

The exponential form of Fourier's integral formula (11) for g is

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(t) \exp(iu(x - t))\, dt \right) du,$$

and this becomes

$$e^{-\gamma x} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\gamma t} f(t)\, e^{iux}\, e^{-iut}\, dt \right) du.$$

Thus

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\gamma + iu)x} \left(\int_{-\infty}^{\infty} e^{-(\gamma + iu)t} f(t)\, dt \right) du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\gamma + iu)x} \left(\int_{0}^{\infty} e^{-(\gamma + iu)t} f(t)\, dt \right) du,$$

since $f(t) = 0$, for t negative.

This gives us a new integral transform,

$$F(\gamma + iu) = \int_{0}^{\infty} e^{-(\gamma + iu)t} f(t)\, dt,$$

with inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\gamma + iu)x} F(\gamma + iu)\, du.$$

This is the *Laplace transform.*

* *Pierre Simon Laplace* (1749–1827) was one of the leading mathematicians of his day. Despite the political changes in France, Laplace (unlike some other mathematicians) was able to continue his mathematical pursuits. (He received honours from both Napoleon and Louis XVIII). He is best remembered for his work on Celestial Mechanics and Probability. He introduced the method of "Laplace transforms" in his analytic study of probability.

It is convenient to write $\gamma + iu = p$ and the variable t is usually preferred to x when using Laplace transforms. With these modifications we make the following definition.

Definition

The function F defined by

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

is called the **Laplace transform** of f; the corresponding **inversion formula** is

$$f(t) = \frac{1}{2\pi i} \int_\Gamma e^{pt} F(p)\, dp,$$

where Γ is the line $\operatorname{Re} p = \gamma$ and where $\operatorname{Re} p$ must be large enough for the integral for F to converge. (Just how large γ must be for convergence, and the manner in which these integrals converge we shall go into in Section 14.2.)

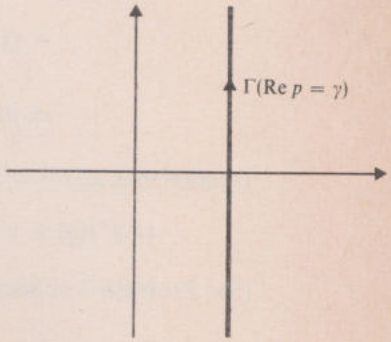


Fig. 5

In some contexts it is useful to denote F the Laplace transform of f by $\mathcal{L}[f]$. We write $\mathcal{L}[f] = F$ and $\mathcal{L}[f](p) = F(p)$.

In most elementary applications it is possible to avoid the use of the inversion integral by building up a catalogue of functions and their transforms in the hope that, having found a transform F as the solution to a problem, we can recognize the appropriate f to which it corresponds. Many such tables are available; their use is extended dramatically by various formulas for dealing with combinations of tabulated functions. (These are discussed in *Unit M201 29, Laplace Transforms.*) Here we shall be interested mainly in the inversion formula, which *has* to be employed when tabulation techniques break down. Before we discuss the inversion formula, we shall calculate a few Laplace transforms in the following examples.

Example 4

If $f(t) = 1$,

$$\mathcal{L}[f](p) = F(p) = \int_0^\infty e^{-pt}\, dt$$

$$= \frac{1}{p}, \quad \operatorname{Re} p > 0.$$

Example 5

If $f(t) = t$,

$$\mathcal{L}[f](p) = F(p) = \int_0^\infty te^{-pt}\, dt$$

$$= \frac{te^{-pt}}{-p}\bigg|_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}\, dt$$

$$= \frac{1}{p^2}, \quad \operatorname{Re} p > 0.$$

Example 6

If $f(t) = e^{at}$, $a \in \mathbf{R}$,

$$\begin{aligned}\mathcal{L}[f](p) = F(p) &= \int_0^\infty e^{at}\, e^{-pt}\, dt \\ &= \left.\frac{e^{(a-p)t}}{a-p}\right|_0^\infty \\ &= \frac{1}{p-a}, \quad \operatorname{Re} p > a.\end{aligned}$$

Example 7

We can find the Laplace transform of f' in terms of that of f when both f and f' are continuous; we get

$$\begin{aligned}\mathcal{L}[f'](p) &= \int_0^\infty f'(t)\, e^{-pt}\, dt \\ &= f(t)\, e^{-pt}\Big|_0^\infty + p\int_0^\infty f(t)\, e^{-pt}\, dt \\ &= pF(p) - f(0), \quad \operatorname{Re} p > 0.\end{aligned}$$

(Notice that $\mathcal{L}[f'] \neq F'$.) Similarly, one can easily show that

$$(\mathcal{L}f'')(p) = p^2F(p) - pf(0) - f'(0), \quad \operatorname{Re} p > 0.$$

(See Problem 2 of Section 14.3.)

Example 8

The previous example shows how the Laplace transform can be used to convert the operation of differentiation to an algebraic operation, and therein lies its use. We shall now exemplify the schema represented in the left-hand figure on page 10. Consider the differential equation

$$f''(t) + 2f'(t) - 3f(t) = 1,$$

where $f(0) = 1$ and $f'(0) = 0$.

Step 1 Transform the Differential Equation

We use the previous example to write down the transforms of f'' and f' in terms of F, the transform of f, and we use Example 6 to transform the right-hand side. We get

$$p^2F(p) - pf(0) - f'(0) + 2pF(p) - 2f(0) - 3F(p) = \frac{1}{p}, \quad \operatorname{Re} p > 0,$$

and so $F(p)$ satisfies the algebraic equation

$$p^2F(p) - p + 2pF(p) - 2 - 3F(p) = \frac{1}{p}.$$

Notice that the differential equation in f has been converted to an algebraic equation in F (and a linear one at that). Notice also that the effectiveness of the method depends on a knowledge of the values of f and f' at zero.

Step 2 Solve the Algebraic Equation

The solution is

$$F(p) = \frac{p^2 + 2p + 1}{p(p^2 + 2p - 3)}$$

with the condition now that $\operatorname{Re} p > 1$, to avoid the zeros of $p \longrightarrow p^2 + 2p - 3$.

Step 3 Invert F

The problem now is to infer the form of f from our knowledge of F, in other words to invert the transform. We have chosen this case to be particularly easy. We can express $F(p)$ in partial fractions.

$$F(p) = -\frac{1}{3p} + \frac{1}{3(p+3)} + \frac{1}{p-1}.$$

Using the fairly obvious result that the transform of a sum, $h + g$, is the sum of the transforms, $H + G$, and using the results of Examples 4 and 6, we deduce that

$$f(t) = -\tfrac{1}{3} + \tfrac{1}{3}e^{-3t} + e^{t}.$$

Self-Assessment Questions

2. Define the Laplace transform of a function f.

3. Compare and contrast the treatments of differential equations exhibited in Examples 3 and 8.

Solutions

2. See page 19.

3. The overall similarities in the methods are shown in the two schematic diagrams on page 10.

   The differences of treatment arise because in Example 3 we are dealing with a partial differential equation for which the boundary conditions are given in terms of the function (to be found), whereas in Example 8 we are dealing with an ordinary differential equation for which the boundary conditions are initial ($t = 0$) conditions on the function (to be found) and its derivative.

Summary

In this section, we have seen some examples of integral transforms. We have seen how the (finite) sine and cosine transform can be generalized to the (infinite) Fourier sine and Fourier cosine transforms or Fourier Transforms and the essence of these transforms is expressed in Fourier's Integral Formula. This formula was used to develop the Laplace transform. Some Laplace transforms can be inverted using tables of Laplace transforms, but if these break down the inversion formula has to be used. This formula involves an integral in the complex plane.

(In this section we have not investigated the conditions which ensure the validity of the inversion formula and we have not considered methods of evaluating the inversion integral. These two matters are taken up in Sections 14.2 and 14.4 respectively.)

14.2 THE INVERSION FORMULA FOR THE LAPLACE TRANSFORM

In this section we shall investigate the conditions under which the inversion formula for the Laplace Transform is valid. But first a preliminary problem.

Preliminary Problem

Let the function F be analytic on a region containing the half-plane $\operatorname{Re} p \geqslant \gamma$. If real constants K and $k(>0)$ exist such that $|F(p)| < |Kp^{-k}|$, show that

$$F(p_0) = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\gamma-ir}^{\gamma+ir} \frac{F(p)}{p_0 - p} dp,$$

where p_0 is such that $\operatorname{Re} p_0 > \gamma$.

The result is known as the *extended form of Cauchy's Formula.*

(Hint: Consider the rectangular contour Γ with vertices $\gamma \pm ir$, $r \pm ir$ where $r > |\gamma|$ and r is so large that p_0 lies inside Γ. See Fig. 6.)

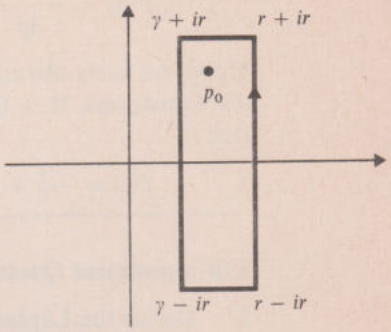


Fig. 6

Solution

By Cauchy's formula, we have

$$\begin{aligned} F(p_0) &= \frac{1}{2\pi i} \int_\Gamma \frac{F(p)}{p - p_0} dp \\ &= \frac{1}{2\pi i} \left(- \int_{\gamma-ir}^{\gamma+ir} \frac{F(p)}{p - p_0} dp + \int_C \frac{F(p)}{p - p_0} dp \right), \end{aligned} \tag{1}$$

where $\Gamma = C + [\gamma + ir, \gamma - ir]$.

If $p \in C$, then

$$\left| \frac{F(p)}{p - p_0} \right| < \frac{|K|}{|p|^k |p - p_0|} = \frac{|K|}{|p|^{k+1} |1 - (p_0/p)|}.$$

Now let r be so large that if $p \in C$, then $|p_0/p| < \frac{1}{2}$. Then

$$|1 - (p_0/p)| > \tfrac{1}{2}.$$

Also, if $p \in C$, $|p| \geqslant r$, and so

$$\left| \frac{F(p)}{p - p_0} \right| < \frac{2|K|}{r^{k+1}}, \quad p \in C.$$

Hence,

$$\int_C \frac{F(p)}{p - p_0} dp < \frac{2|K|}{r^{k+1}} (4r - 2\gamma), \text{ since } 4r - 2\gamma \text{ is length of } C,$$

and so

$$\lim_{r\to\infty} \int_C \frac{F(p)}{p - p_0} dp = 0.$$

From (1), we have

$$F(p_0) = \lim_{r\to\infty} \frac{-1}{2\pi i} \int_{\gamma-ir}^{\gamma+ir} \frac{F(p)}{p - p_0} dp$$

and so

$$F(p_0) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\gamma-ir}^{\gamma+ir} \frac{F(p)}{p_0 - p} dp, \quad \text{as required.}$$

We have already seen, in Problem 5 of Section 11.8 of *Unit 11*, that if real constants C and γ can be chosen so that $|f(t)| \leqslant Ce^{\gamma t}$ when $t \geqslant 0$, then the Laplace Transform of f, F, is analytic on the half-plane $\operatorname{Re} p > \gamma$. As we shall see this fact gives us an alternative formal derivation of the inversion integral to that offered in the previous section. We know from the preliminary problem above, that if a function F is analytic on a half-plane $\operatorname{Re} p \geqslant \gamma$ and constants K and k

(with $k > 0$) can be found such that $|F(p)| < |Kp^{-k}|$, then the extended form of Cauchy's formula applies and we can write

$$F(p) = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\gamma-ir}^{\gamma+ir} \frac{F(z)}{p-z} dz. \tag{2}$$

If we define an *inverse Laplace Transform* $\mathcal{L}^{-1}$ as an operation for which $\mathcal{L}^{-1}[F] = f$ then, applying this (without formal justification) to each side of (2), we get

$$\mathcal{L}^{-1}[F](t) = f(t) = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\gamma-ir}^{\gamma+ir} F(z)\mathcal{L}^{-1}\left[\frac{1}{p-z}\right] dz,$$

where $\mathcal{L}^{-1}\left[\frac{1}{p-z}\right]$ means the value at t of $\mathcal{L}^{-1}\left[p \longrightarrow \frac{1}{p-z}\right]$. But the result of Example 6 of Section 14.1 tells us that $\mathcal{L}^{-1}\left[\frac{1}{p-z}\right]$ is e^{zt} and so we deduce that

$$f(t) = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\gamma-ir}^{\gamma+ir} e^{zt}F(z)\, dz,$$

which is the inversion integral that we obtained previously (page 18).

Since it is off our main line of interest we have made no attempt to justify this argument (which involves interchanging the order of a limit and an integral), but you should notice that the extended form of Cauchy's Formula applies only when F satisfies a condition

$$|F(p)| < |Kp^{-k}|, \quad k > 0.$$

Whatever other conditions we may be obliged to lay down later, it looks as though this must hold, and this is a condition on F, the transform.

When developing our equally cavalier approach in the previous section, using the Fourier integral, we noted, on page 18, that if f satisfies the condition $|f(t)| < |Ce^{\alpha t}|$, and $f(t) = 0$ for $t < 0$, then the inversion integral holds. This is a condition on the original function, so let us now ask what sort of conditions must be satisfied for the inversion integral to hold? Are they conditions on f, or on the transform F, or on both? In fact, we can find sufficient conditions on either of these functions, each set of conditions having its use. Conditions on F are easier to use—after all this is the function that we want to plug into the inversion integral. On the other hand, the conditions on f are less restrictive than those on F and they can often be checked by direct reference to the problem to be solved. Although we do not, of course, have explicit information about the solution to a problem before it is solved, we can often deduce general characteristics of the solution directly from the nature of the problem.

We shall in fact deal *only* with the transform function F, and show that the condition above is sufficient to ensure that the inversion integral exists and gives the correct answer. We shall do this in a chain of theorems; first of all we deal with the existence of the integral. (You should be able to follow the proofs—some of the ideas are used later—but you will *not* be expected to be able to reproduce them. Some real analysis ideas and results are used which you may not have come across; these are listed on page 71.)

Theorem 1

Let the function F satisfy the following conditions:

(i) F is analytic on a region containing the half-plane $\operatorname{Re} p \geqslant x_0$;

(ii) there exist real numbers $k > 1$ and K such that $|F(p)| < K|p^{-k}|$, for p in $\{p: \operatorname{Re} p \geqslant x_0\}$ and $|p|$ greater than some $|p_0|$;

(iii) $F(p)$ is real when p is real.

Then $\frac{1}{2\pi i} \lim_{r\to\infty} \int_{\gamma-ir}^{\gamma+ir} e^{pt}F(p)\, dp$, $\gamma \geqslant x_0$, exists and defines a continuous function (of t).

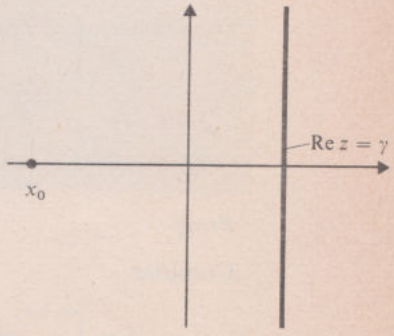


Fig. 7

Note You may have noticed that the condition $k > 0$, required earlier for the application of Cauchy's Formula, has now become $k > 1$. In fact, Theorems 1, 2, 3, 4 and 5 can be proved for $k > 0$, provided additional conditions are introduced, but the proofs are more difficult.

Proof

On the contour for the integral $\int_{\gamma-ir}^{\gamma+ir} e^{pt}F(p)\,dp$, we have $p = \gamma + iy$, with γ fixed, and so we can write

$$\int_{\gamma-ir}^{\gamma+ir} e^{pt}F(p)\,dp = ie^{\gamma t}\left(\int_{-r}^{0} e^{iyt}F(\gamma + iy)\,dy + \int_{0}^{r} e^{iyt}F(\gamma + iy)\,dy\right)$$
$$= ie^{\gamma t}\int_{0}^{r}\left(e^{-iyt}F(\gamma - iy) + e^{iyt}F(\gamma + iy)\right)dy.$$

Since $F(p)$ is real on the real axis, we may use the Schwarz Reflection Principle (*Unit 11*) to deduce that $F(\bar{p}) = \overline{F(p)}$ and hence to reduce the integrand to

$$2\,\mathrm{Re}\left(e^{iyt}F(\gamma + iy)\right) = \phi(t, y), \quad \text{say.}$$

Then the inversion integral takes the form

$$\frac{1}{2\pi i}\lim_{r\to\infty} ie^{\gamma t}\int_{0}^{r}\phi(t, y)\,dy.$$

Now from condition (ii) on F, there must be an M and a y_0 such that

$$2|e^{iyt}F(\gamma + iy)| = 2|F(\gamma + iy)| < \frac{M}{(\gamma^2 + y^2)^{k/2}}, \quad |y| > |y_0|,$$

$$\leqslant \frac{M}{|y|^k}.$$

Thus

$$|\phi(t, y)| \leqslant \frac{M}{|y|^k}, \quad |y| \geqslant |y_0|,$$

where M is independent of t.

We can now use the upper bound for $|\phi(t, y)|$ to show that the integral converges uniformly (with respect to t): see item 2 in the Appendix. Further, since $t \longrightarrow \phi(t, y)$ is a continuous function for all values of y, we can deduce, from item 3 in the Appendix, that the integral defines a continuous function f, where

$$f(t) = \frac{1}{2\pi i}\lim_{r\to\infty}\int_{\gamma-ir}^{\gamma+ir} e^{pt}F(p)\,dp,$$

which we usually write as

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)\,dp. \quad \blacksquare$$

Theorem 2

The function f of Theorem 1, that is

$$f: t \longrightarrow \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)\,dp$$

is independent of γ.

Proof

Consider

$$\lim_{r\to\infty}\int_{\gamma_1-ir}^{\gamma_1+ir} e^{pt}F(p)\,dp,$$

where $\gamma_1 > \gamma \geqslant x_0$. Consider further the rectangular contour Γ (shown in Fig. 8) which has vertices $\gamma_1 \pm ir$, $\gamma \pm ir$.

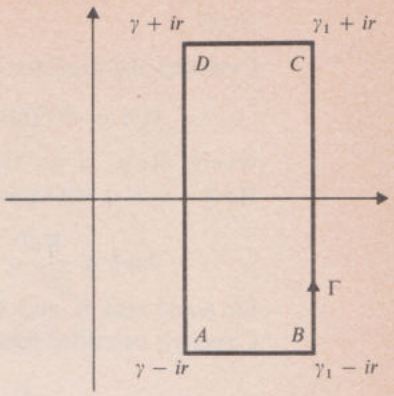


Fig. 8

Since F is analytic inside Γ we have $\int_\Gamma e^{pt}F(p)\,dp = 0$. If we can show that the integrals along AB and CD each approach zero as r increases then we can show that as r increases the sum of the integrals along DA and BC approaches zero, and the result is proved.

We have

$$\int_{CD} e^{pt}F(p)\,dp = \int_{\gamma_1}^{\gamma} e^{xt}e^{irt}F(x + ir)dx, \quad \text{where } x = \operatorname{Re} p.$$

Since $|F(p)| < \dfrac{M}{|p|^k}$, we have

$$\int_{CD} e^{pt}F(p)\,dp < \frac{M}{|p|^k}\int_{\gamma_1}^{\gamma} e^{xt}\,dx,$$

and so has limit zero as $|p|$ increases. The side AB may be dealt with in a similar way, and the result follows. ■

The next theorem gives us a bound on f.

Theorem 3

Under the same conditions as in Theorem 1, there exist real constants C and x_0 such that

$$|f(t)| = \left| \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}F(p)\,dp \right| \leqslant C\exp(x_0 t).$$

Proof

As in the proof of Theorem 1, let $\phi(t, y) = 2\operatorname{Re}\left(e^{iyt}F(\gamma + iy)\right)$, where $\gamma + iy = p$. We have

$$|f(t)| = \frac{e^{\gamma t}}{2\pi}\int_0^{\infty} |\phi(t, y)|\,dy.$$

But

$$\begin{aligned} |\phi(t, y)| &= 2|\operatorname{Re}\left(e^{iy}F(\gamma + iy)\right)| \\ &\leqslant 2|e^{iy}F(\gamma + iy)| \\ &= 2|F(\gamma + iy)|. \end{aligned}$$

Thus

$$|f(t)| \leqslant \frac{e^{\gamma t}}{\pi}\int_0^{\infty} |F(\gamma + iy)|\,dy.$$

But, from Theorem 2, we know that $f(t)$ is independent of γ for $\gamma \geqslant x_0$. We can therefore take $\gamma = x_0$, giving

$$|f(t)| \leqslant C\exp(x_0 t)$$

with $C = \dfrac{1}{\pi}\displaystyle\int_0^{\infty} |F(x_0 + iy)|\,dy$. ■

Theorem 4

The function f of Theorem 1, that is

$$f: t \longrightarrow \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}F(p)\,dp$$

is such that $f(t) = 0$ for $t < 0$.

Proof

Consider the function h, where

$$h(p) = e^{pt}F(p)(p - p_1)$$

where $\operatorname{Re} p_1 > \gamma$. This function is analytic on $\{p:\operatorname{Re} p \geqslant \gamma\}$. Because $\operatorname{Re} p - \gamma \geqslant 0$, we know that if $t < 0$ then $\operatorname{Re} pt \leqslant \gamma t$. Then $|e^{pt}| \leqslant e^{\gamma t}$, and so

$$|h(p)| \leqslant \frac{Ke^{\gamma t}}{|p|^{k-1}}$$

for some real K and $k > 1$. For $t < 0$, h therefore satisfies the conditions of Cauchy's extended formula and we can write

$$\begin{aligned} h(p_1) &= -\frac{1}{2\pi i}\lim_{r\to\infty}\int_{\gamma-ir}^{\gamma+ir}\frac{h(p)}{p-p_1}\,dp \\ &= -\frac{1}{2\pi i}\lim_{r\to\infty}\int_{\gamma-ir}^{\gamma+ir}e^{pt}F(p)\,dp \\ &= -f(t). \end{aligned}$$

But $h(p_1) = 0$, and so $f(t) = 0$ for $t < 0$. ■

The last theorem in this section tells us that the function f defined by the inversion integral is indeed a function whose Laplace transform is F.

Theorem 5

If $f:t \longrightarrow \dfrac{1}{2\pi i}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)\,dp$, then

$$\int_0^\infty e^{-pt}f(t)\,dt = F(p).$$

Proof

We have

$$f(t) = \frac{1}{2\pi i}\lim_{r\to\infty}\int_{\gamma-ir}^{\gamma+ir}e^{zt}F(z)\,dz = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{zt}F(z)\,dy$$

where $z = \gamma + iy$. We use z rather than p, because we want to use p as the variable in the transform of f, which is

$$\begin{aligned} \int_0^\infty e^{-pt}f(t)\,dt &= \lim_{T\to\infty}\int_0^T e^{-pt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{zt}F(z)\,dy\right)dt \\ &= \frac{1}{2\pi}\lim_{T\to\infty}\int_{-\infty}^{\infty}F(z)\left(\int_0^T e^{t(z-p)}\,dt\right)dy, \end{aligned} \tag{3}$$

where the interchange of the order of integration is justified by the uniform convergence (with respect to t) of the inversion integral. (See Appendix item 1.)

Now, from Example 6 of Section 14.1,

$$\lim_{T\to\infty}F(z)\int_0^T e^{t(z-p)}\,dt = \frac{F(z)}{p-z}, \quad \operatorname{Re} p > \operatorname{Re} z,$$

and so, if we can take the limit operation inside the integral with respect to y in (3), we get

$$\begin{aligned} \int_0^\infty e^{-pt}f(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(z)}{p-z}\,dy \\ &= F(p), \quad \text{by the extended form of Cauchy's Formula.} \end{aligned}$$

It remains to justify taking the limit operation inside the first integral. To do this we have to show two things. We have to show that

$$F(z)\int_0^T e^{t(z-p)}\,dt$$

approaches its limit uniformly with respect to y. We also have to show that the improper integral

$$\int_{-\infty}^{\infty} F(z)\left(\int_0^T e^{t(z-p)}\,dt\right)dy$$

converges uniformly with respect to T.

Now

$$\int_0^T e^{t(z-p)}\,dt = \frac{1-e^{-T(p-z)}}{p-z},$$

and so, from (3),

$$\int_0^{\infty} e^{-pt}f(t)\,dt = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{F(z)}{p-z}(1-e^{-T(p-z)})\,dy, \quad z=\gamma+iy. \tag{4}$$

We now show that this limit is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(z)}{p-z}\,dy, \quad z=\gamma+iy$$

(in other words, we can take the limit operation inside the integral).

We can apply the extended form of Cauchy's Formula to this integral to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(z)}{p-z}\,dy = F(p). \tag{5}$$

To prove that this is indeed the limit we can proceed as follows. From (4) and (5), we obtain

$$\left|\int_0^{\infty} e^{-pt}f(t)\,dt - F(p)\right| = \lim_{T\to\infty}\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(z)}{p-z}e^{-T(p-z)}\,dy\right|.$$

Since $\operatorname{Re} z = \gamma$,

$$\left|\int_{-\infty}^{\infty}\frac{F(z)}{p-z}e^{-T(p-z)}\,dy\right| \leqslant \int_{-\infty}^{\infty}\left|\frac{F(z)}{p-z}\right|e^{-T(\operatorname{Re}p-\gamma)}\,dy$$

$$= e^{-T(\operatorname{Re}p-\gamma)}\int_{-\infty}^{\infty}\left|\frac{F(z)}{p-z}\right|dy.$$

Because of condition (ii) of Theorem 1, this last integral exists and so the right-hand side has limit zero as T increases because $\operatorname{Re} p > \gamma$. ■

This proof of Theorem 5 completes the set of theorems on the conditions on the transform F, for the inversion integral to exist and to reinstate the original function. We have proved sufficiency of these conditions; they are not the least restricting set, but they are simple to test for and have a wide enough applicability to be useful for practical work.

If you are given an inversion integral to evaluate, it is not always necessary for you to check its validity. Sometimes it is just as quick to work out the integral and then take the Laplace transform of the answer. If this yields the correct function, all is well. On the other hand, the evaluation of the Laplace transform is sometimes more difficult than testing that the conditions of Theorem 1 are satisfied, especially, for example, in cases where $f(t)$ is given by an infinite series.

As we have mentioned, conditions can also be obtained on the function f itself. A proof for these conditions amounts essentially to a proof of Fourier's integral formula, from which we derived the inversion integral. This is really a problem in real analysis, and so we shall not tackle it here (see, for example,

Titchmarsh, E. C. *The Theory of Functions*, second edition, Oxford University Press, 1939).

Summary

We have proved a chain of five theorems on the validity of the inversion integral. You should have studied these to the depth required for you to understand the proofs, because some of the techniques used are useful in other contexts. However, you will not be expected to reproduce the proofs. You should try to remember the conditions given in Theorem 1, for the validity of the inversion integral; you may be expected to test given functions for these conditions.

Self-Assessment Questions

1. State a set of sufficient conditions on the function F for the integral

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)\,dp, \quad \gamma \geqslant x_0,$$

to exist and define a continuous function.

2. Classify the following statements as True or False, given that the conditions of Theorem 1 are satisfied.

(a) If $\gamma \neq \sigma$, and $\gamma \geqslant x_0, \sigma \geqslant x_0$, then

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)\,dp \neq \int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}F(p)\,dp.$$

(b) Given x_0 and C, then

$$|f(t)| \leqslant C\exp(x_0 t) \quad \text{for all } t.$$

(c) $f(t)$ is negative if t is negative.

(d) Theorem 5 states that

$$\mathcal{L}[f] = F.$$

Solutions

1. See Theorem 1.

2. (a) False: see Theorem 2.

(b) False: see Theorem 3.

(c) False: see Theorem 4.

(d) True: see Theorem 5.

14.3 PROBLEMS

1. (a) If f is continuous on a finite closed interval $[a, b]$, where $a, b > 0$, and if $f(t) = 0$, $t \notin [a, b]$, show that F, the Laplace transform of f, is an entire function.
(Hint: See the paragraph following the preliminary problem of Section 14.2.)

   (b) Verify this result using the function

$$f(t) = \begin{cases} 1, & t \in [1, 2] \\ 0, & \text{otherwise.} \end{cases}$$

2. If F is the Laplace transform of f, show that

   (a) the Laplace transform of f'' is $p^2F(p) - pf(0) - f'(0)$, $\operatorname{Re} p > 0$.

   (b) $\lim_{p \to 0^+} pF(p) = \lim_{t \to \infty} f(t)$, where p is real.

3. In *Unit 11*, we proved that if real constants C and γ can be found such that $|f(t)| \leqslant |Ce^{\gamma t}|$, $t \geqslant 0$, then the Laplace transform is analytic on the half-plane $\operatorname{Re} p > \gamma$. If x_0 is the greatest lower bound of the set of γ's for which such C's can be found, then the line $\operatorname{Re} p = x_0$ is called the *abscissa of convergence* of the Laplace transform F and $\{p : \operatorname{Re} p > x_0\}$ is the "largest" half-plane on which F exists and F is analytic on that half-plane.

   (a) Show that if F is analytic on the half-plane $\operatorname{Re} p > \gamma$ and F has a pole on $\operatorname{Re} p = \gamma$, then $\gamma = x_0$.

   (b) (i) If $f(t) = \sin at$, $a \in \mathbf{C}$, find F and show that the abscissa of convergence is the line $\operatorname{Re} p = \operatorname{Im} a$.

   (ii) Find the "largest" half-plane on which the Laplace transform of $t \longrightarrow \sinh 3t$ exists.

4. (i) Show that the function $F(p) = \dfrac{1}{p^2 - 1}$ satisfies the conditions of Theorem 1 (page 23).

   (ii) Let $F(p) = \dfrac{p}{1 + p^2}$. Show that there exist real numbers $k > 0$ and K such that $|F(p)| < K|p^{-k}|$, for p in $\{p : \operatorname{Re} p \geqslant x_0\}$ and $|p|$ greater than some p_0. (See the note at the top of page 24.)

Solutions

1. (a) Since f is continuous on $[a, b]$ it is bounded on $[a, b]$, by A say. Then, given γ,

$$|f(t)| \leqslant Ce^{\gamma t}, \quad t \in [a, b],$$

where

$$C = A \cdot \inf\{e^{\gamma t}, t \in [a, b]\}.$$

Since $[a, b]$ is finite, C exists for any choice of γ. If $t \notin [a, b]$, then $|f(t)| = 0$ and whatever γ is, any positive value of C will suffice. Thus F is entire by Problem 5 of Section 11.8 of *Unit 11*.

   (b) $F(p) = \displaystyle\int_1^2 e^{-pt}\,dt = \frac{e^{-p} - e^{-2p}}{p}$, which is entire.

2. (a) $\displaystyle\int_0^\infty e^{-pt}f''(t)\,dt = e^{-pt}f'(t)\Big|_0^\infty + p\int_0^\infty e^{-pt}f'(t)\,dt$, by integration by parts, (if f' unbdd !)

$$= -f'(0) + p\left(e^{-pt}f(t)\Big|_0^\infty + p\int_0^\infty e^{-pt}f(t)\,dt\right), \quad \text{if } \operatorname{Re} p > 0,$$

$$= p^2F(p) - pf(0) - f'(0).$$

(b) From Example 7 of Section 14.1, we have

$$\lim_{p\to 0^+} pF(p) = \lim_{p\to 0^+} \int_0^\infty e^{-pt} f'(t)\,dt + f(0)$$

$$= \int_0^\infty f'(t)\,dt + f(0)$$

$$= \lim_{t\to\infty} f(t) - f(0) + f(0) = \lim_{t\to\infty} f(t).$$

3. (a) From the definition, we know that $x_0 \leqslant \gamma$. If $x_0 < \gamma$, then there would be a pole in the half-plane $\operatorname{Re} p > x_0$ and F would not be analytic on that half-plane, and so $\gamma = x_0$.

(b) (i) $$F(p) = \int_0^\infty e^{-pt} \sin at\,dt$$

$$= \frac{a}{p^2 + a^2}, \quad \text{after some calculation.}$$

The function F has poles at $p = \pm ia$, at which points $\operatorname{Re} p = \pm \operatorname{Im} a$, and so the abscissa of convergence is the line $\operatorname{Re} p = \operatorname{Im} a$.

(ii) $\{p : \operatorname{Re} p > 3\}$ $\left(\text{since } \int_0^\infty e^{-pt} \sinh 3t\,dt = \frac{3}{p^2 - 9}\right)$.

4. (i) The function F has poles at ± 1, so we can take x_0 to be any real number greater than 1. Thus condition (i) is satisfied. Also,

$$|F(p)| = \frac{1}{|p^2 - 1|} \leqslant \frac{1}{|p^2| - 1}$$

$$\leqslant \frac{1}{|p^2| - |p^2/2|}, \quad \text{if } |p^2| > 2,$$

$$= \frac{2}{|p^2|}.$$

Thus condition (ii) is satisfied with $p_0 = \sqrt{2}$, $k = 2$ and $K = 2$. Condition (iii) is also satisfied.

(ii) We have

$$|F(p)| = \frac{|p|}{|1 + p^2|} \leqslant \frac{|p|}{|p^2| - 1}, \quad \text{if } |p^2| > 1,$$

$$\leqslant \frac{1}{|p|}, \quad \text{if } |p^2| > 1.$$

Thus the condition is satisfied with $p_0 = 1$, $k = 1$ and $K = 1$.

14.4 EVALUATION OF THE INVERSION INTEGRAL

The evaluation of the inversion integral

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)\,dp$$

does of course depend on the form of the transform function F. But in each case the contour is similar, and we also know that F is indeed a Laplace transform and so has certain properties; in particular, we often know something about its behaviour far from the origin. As a consequence, it is possible to deal with large classes of inversion integrals using one technique and the decision on whether the technique is applicable in a particular case rests on a knowledge of the behaviour of the transform F for large values of $|p|$.

To be more precise, in each case the contour is the line $\operatorname{Re} p = \gamma$ and we evaluate the integral using the techniques of *Unit 10*, based on the Residue Theorem. We construct a closed contour, part of which consists of a segment of the line $\operatorname{Re} p = \gamma$, and along which we can integrate $p \longrightarrow e^{pt}F(p)$, using the Residue Theorem. The idea is to show that as the contour is increased in size, such that the line segment becomes the complete line $\operatorname{Re} p = \gamma$, the integral along the remaining curved part of the contour approaches zero.

There are various ways of constructing the closed contour. Perhaps the most useful, and one with which you are familiar from *Unit 10*, is to use an arc of a circle.

Theorem 6

Let Γ_1 and Γ_2 be the contours

$$\Gamma_1 = \left\{p = re^{i\theta}: \frac{\pi}{2} - \alpha \leqslant \theta \leqslant \pi, \sin\alpha = \frac{\gamma}{r}, \gamma > 0\right\}$$

and

$$\Gamma_2 = \left\{p = re^{i\theta}: -\pi \leqslant \theta \leqslant \alpha - \frac{\pi}{2}, \sin\alpha = \frac{\gamma}{r}, \gamma > 0\right\}.$$

If there exist positive constants K, k and p_0 such that on the contours Γ_1 and Γ_2 the function F satisfies

$$|F(p)| < K|p|^{-k}, \quad |p| > p_0,$$

then, for $t > 0$,

$$\lim_{r\to\infty}\int_{\Gamma_1} F(p)e^{pt}\,dp = \lim_{r\to\infty}\int_{\Gamma_2} F(p)e^{pt}\,dp = 0.$$

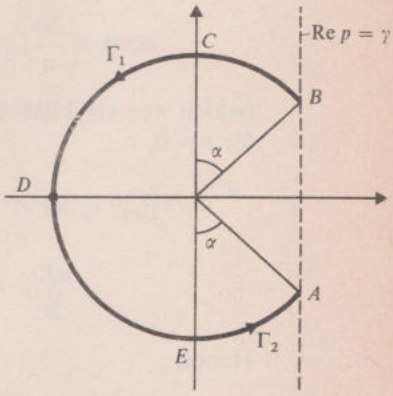


Fig. 9

Notice that this theorem places no further restrictions on F than does Theorem 1 (for the validity of the inversion integral). But notice the condition $t > 0$, see Example 2 below.

Proof

(We have chosen to prove this theorem for each of the contours $\Gamma_1(BCD)$ and $\Gamma_2(DEA)$—see Fig. 9—separately rather than for the complete contour $\Gamma_1 + \Gamma_2(BCA)$. This is a more useful form because it can be applied to functions for which we may have to cut the plane—if the cut is made along the negative real axis.) Consider the integral along BC of $p \longrightarrow F(p)e^{pt}$: denote it by I_{BC}.

$$|I_{BC}| = \left|\int_{\pi/2-\alpha}^{\pi/2} F(p)e^{rt\exp(i\theta)}re^{i\theta}\,d\theta\right|$$

$$\leqslant K\int_{\pi/2-\alpha}^{\pi/2} |p|^{-k}e^{rt\cos\theta}r\,d\theta, \quad \text{if } r > p_0,$$

$$\leqslant Kr^{1-k}\int_{\pi/2-\alpha}^{\pi/2} e^{rt\cos\theta}\,d\theta.$$

Now, since $\pi/2 - \alpha \leqslant \theta \leqslant \pi/2$ and $\sin\alpha = \gamma/r$, we have

$$0 \leqslant \cos\theta \leqslant \sin\alpha = \frac{\gamma}{r}.$$

Thus, because $t > 0$,

$$|I_{BC}| \leqslant Kr^{1-k}e^{t\gamma}\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right]$$
$$= Kr^{1-k}e^{t\gamma}\alpha.$$

Since $r = \gamma/\sin\alpha$,

$$|I_{BC}| \leqslant K\left(\frac{\gamma}{\sin\alpha}\right)^{1-k} e^{t\gamma}\alpha.$$

As r gets large, α approaches 0, and so

$$\lim_{r\to\infty} |I_{BC}| \leqslant \lim_{\alpha\to 0} K\gamma^{1-k}e^{t\gamma}\left(\frac{\alpha}{\sin\alpha}\right)^{1-k}\alpha^k$$
$$= 0, \quad \text{since } \lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha} = 1.$$

For the integral along CD we have

$$|I_{CD}| \leqslant Kr^{-k}\int_{\pi/2}^{\pi} e^{tr\cos\theta} r\, d\theta$$
$$= Kr^{1-k}\int_0^{\pi/2} e^{-tr\sin\phi}\, d\phi, \quad \text{substituting } \phi = \theta - \pi/2.$$

We can now apply the inequality,

$$\sin\phi \geqslant \frac{2\phi}{\pi}, \quad \text{if } 0 \leqslant \phi \leqslant \frac{\pi}{2}$$

(which was established in Problem 1 of Section 10.5 of *Unit 10*), and obtain, for $t > 0$,

$$|I_{CD}| \leqslant Kr^{1-k}\int_0^{\pi/2} e^{-2tr\phi/\pi}\, d\theta$$
$$= \frac{\pi K}{2t}r^{-k}(1 - e^{-tr}).$$

Hence,

$$\lim_{r\to\infty} |I_{CD}| = 0.$$

This proves the result for the contour $\Gamma_1(BCD)$. The proof for $\Gamma_2(DEA)$ is similar. ■

We have proved Theorem 6 only for contours with a circular arc. For some functions, however, it is difficult to work out K, k and p_0 for circular arcs. Other contours can be used in these cases, the most common being rectangular arcs and parabolic arcs. We shall not tackle these, although the principles of the proofs are similar to those of Theorem 6.

Example 1

Use the inversion integral to find the function f whose Laplace transform F is given by

$$F(p) = \frac{1}{p^2 - 1}.$$

If the conditions of Theorem 1 hold the inversion integral is valid: under these circumstances the condition of Theorem 6 also holds.

The singularities of F are at ± 1 and so we can take x_0 in Theorem 1 to be any number greater than 1. The function F also satisfies property (ii) of Theorem 1 (see Problem 4 of 14.3). Property (iii) is satisfied because $F(p)$ is real when p is real. Thus

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}}{p^2-1}\,dp, \quad \text{where } \gamma > 1.$$

Consider the closed contour Γ made up of a segment of the line $\operatorname{Re} p = \gamma$ and an arc of a circle, with centre the origin and radius $r > 1$ (Fig. 10).

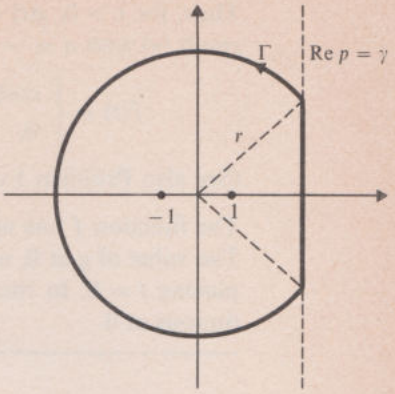


Fig. 10

The residue of $p \longrightarrow e^{pt}/(p^2-1)$ at 1 is $e^t/2$ and at -1 it is $e^{-t}/-2$. Thus

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{e^{pt}}{p^2-1}\,dp = \frac{e^t}{2} - \frac{e^{-t}}{2} = \sinh t.$$

We now apply Theorem 6 with $F(p) = 1/(p^2-1)$, giving

$$f(t) = \sinh t.$$

Example 2

Find a function f whose Laplace transform F is given by

$$F(p) = \frac{pe^{-p}}{1+p^2}, \quad \operatorname{Re} p > 0.$$

Functions such as these, involving an exponential factor, can be tackled as follows.

Suppose

$$g(t) = \int_{\gamma-i\infty}^{\gamma+i\infty} G(p)e^{pt}\,dp.$$

Then

$$\int_{\gamma-i\infty}^{\gamma+i\infty} G(p)e^{ap}e^{pt}\,dp = \int_{\gamma-i\infty}^{\gamma+i\infty} G(p)e^{p(t+a)}\,dp = g(t+a). \qquad (*)$$

Thus the effect of the factor e^{ap} in the Laplace transform is to translate the t scale an amount $-a$.

In this particular case, we take $G(p) = p/(1+p^2)$ and $a = -1$. Let us *assume* that the inversion integral exists. (This is a reasonable assumption in view of Problem 4(ii) of Section 14.3 You can check that the assumption is valid by showing that the Laplace transform of the resulting function is $p \longrightarrow p/(1+p^2)$.) Thus

$$g(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{pe^{pt}}{1+p^2}\,dp, \quad \text{where } \gamma > 0.$$

Consider the closed contour Γ made up of a segment of the line $\operatorname{Re} p = \gamma$ and an arc of a circle with centre the origin and radius r, such that Γ encloses $\pm i$ (Fig. 11) the singularities of the integrand.

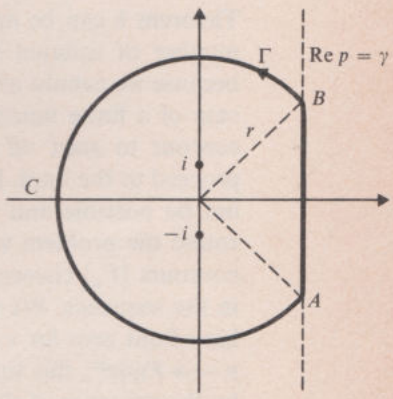


Fig. 11

The residue of $p \longrightarrow pe^{pt}/(1+p^2)$ at i is $e^{it}/2$ and the residue at $-i$ is $e^{-it}/2$. Thus

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{pe^{pt}}{1+p^2}\,dp = \frac{2\pi i}{2\pi i}\left(\frac{e^{it}}{2} + \frac{e^{-it}}{2}\right), \quad \text{by the Residue Theorem.}$$

We now apply Theorem 6 and note that as r increases, the integral along the circular part of the contour, BCA (Fig. 11), approaches zero if $t > 0$.

Thus, for $t > 0$, $g(t) = \cos t$. For $t < 0$, we have $g(t) = 0$. Thus, applying the result (*) with $a = -1$, we have

$$f(t) = \begin{cases} \cos(t-1), & t > 1 \\ 0, & t < 1. \end{cases}$$

(See also Problem 1 of Section 14.5.)

The function f has not been defined at 1 because g has not been defined at 0. The value of g at 0, may, of course, be obtained from the inversion integral by placing $t = 0$. In this case (as you can check), $g(0) = \frac{1}{2}$ and so g is not continuous at 0.

The Representation of the Inversion Integral as a Sum of Residues

The application of Theorem 6 in Examples 1 and 2 shows a general and useful principle in the evaluation of inversion integrals.

(i) Check that the conditions of Theorem 1 (or equivalent) are satisfied.

(ii) Locate the singularities of $p \longrightarrow F(p)e^{pt}$ and calculate the residues.

(iii) Enclose the residues by a contour, Γ, made up of a segment of the line $\operatorname{Re} p = \gamma$ (which is to the right of all singularities of F) and an arc of a circle centre the origin.

(iv) The value of

$$\frac{1}{2\pi i}\int_{\Gamma} e^{pt}F(p)\,dp$$

is the sum of the residues calculated in (ii). (The factor $1/(2\pi i)$ cancels the $2\pi i$ required by the Residue Theorem.)

(v) Check that the conditions of Theorem 6 are satisfied, paying attention to the values of t that are applicable (see Example 2).

(vi) Apply Theorem 6. The inversion integral is the sum calculated in (iv).

(*Note* Although we have not proved it, if the function F in Theorem 6 is a rational function with numerator of lower degree than the denominator, then F will satisfy the conditions of the theorem.)

Theorem 6 can be applied also to cases where the function F has an infinite number of isolated singularities. A little technical adjustment is necessary, because we cannot allow any of the singularities to lie on the contour Γ. In the case of a finite number of singularities this is easy: you just arrange for the contour to start off large enough to enclose all the singularities and then proceed to the limit. For an infinite number of isolated singularities this would not be possible and as Γ became larger it would cross singularities. We get round the problem with a slight modification; we take a discrete sequence of contours $\{\Gamma_n\}$ chosen so that there are no singularities on any of the contours in the sequence. We again get the result that, provided the integrals along Γ_n have limit zero for n large, the inversion integral is the sum of the residues of $p \longrightarrow F(p)e^{pt}$; this time it is an infinite sum, whose convergence is guaranteed by the existence of the inversion integral.

Example 3

Assuming that the inversion integral is valid and can be represented as a sum of residues, find the function f whose Laplace transform is $p \longrightarrow 1/(p^2 \cosh p)$. The singularities of $p \longrightarrow e^{pt}/(p^2 \cosh p)$ occur where

$$p^2 \cosh p = 0.$$

They are therefore at 0 and the points p where $\cos ip = 0$, that is

$$p = \frac{(2n-1)\pi i}{2}, n \in \mathbf{N}.$$

To calculate the residues we need to know the nature of the singularities. The singularity at 0 is a pole of order 2, because cosh 0 is finite and non-zero. The derivative of cosh p is sinh p, which is non-zero at the other singularities, which are therefore simple poles.

From the formula on page 17 of *Unit 10*, the residue at 0 is

$$\lim_{z\to 0}\left[\frac{d}{dz}\left(\frac{e^{pt}}{p^2\cosh p}\cdot p^2\right)\right] = t.$$

By Theorem 2 of *Unit 10*, the residue at $\dfrac{(2n-1)\pi i}{2}$ is

$$\frac{e^{(2n-1)\pi it/2}}{\dfrac{-(2n-1)^2\pi^2}{4}\sinh\big((2n-1)\pi i/2\big)} = \frac{4e^{(2n-1)\pi it/2}}{-(2n-1)^2\pi^2 i\sin\big((2n-1)\pi/2\big)}$$

$$= \frac{4i(-1)^n e^{i(2n-1)\pi t/2}}{(2n-1)^2\pi^2}.$$

The sum of the residues is thus

$$t + \sum_{n=-\infty}^{\infty}\frac{4i(-1)^n\exp\big(i(2n-1)\pi t/2\big)}{(2n-1)^2\pi^2}.$$

Writing

$$\exp\big(i(2n-1)\pi t/2\big) = \cos\big((2n-1)\pi t/2\big) + i\sin\big((2n-1)\pi t/2\big)$$

and recalling that

$$\sin(-\theta) = -\sin\theta \quad \text{and} \quad \cos(-\theta) = \cos\theta,$$

we see that the cosine terms in the sum cancel and the sine terms double up.

(Note, for example, that the term in $\cos(5\pi t/2)$ is obtained from $n = 3$, and the term in $\cos(-5\pi t/2)$ is obtained from $n = -2$. Since $(-1)^3 = -(-1)^{-2}$, these terms cancel.)

We obtain

$$f(t) = t + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\left(\frac{2n-1}{2}\right)\pi t.$$

Example 4 (The Heaviside Expansion Theorem)

This theorem is named after Oliver Heaviside (1850–1925) who was the first person to use the Laplace transform to any extensive degree to solve practical problems. He used his "operational calculus" to solve many important problems in the transmission of electrical signals. His mathematical approach was brusque, to say the least, and unfortunately his work was scorned by mathematicians of the day. It was not until Bromwich and Carson in the 1920's used complex variable techniques to justify Heaviside's results that the true value of his work began to be realized. A result that Heaviside considered to be of particular value was his Expansion Theorem (see Heaviside, O, *Electromagnetic Theory*, 3 vols., 1893–1912).

Because the Laplace transform of $t \longrightarrow e^{at}$ is $p \longrightarrow \dfrac{1}{p-a}$, Heaviside deduced that the inverse Laplace transform of a rational function can be calculated by expressing it in partial fractions and inverting term by term. By considering the way in which partial fractions are calculated, he noted that if $F(p) = r(p)/q(p)$, where r and q are polynomials such that q has a higher degree than r, then the term in the inverse transform corresponding to a non-repeated factor $p - a$ of q is $\dfrac{r(a)}{q'(a)}e^{at}$. He then deduced that if all the factors of q are linear and not

repeated then the inverse transform of F is

$$t \longrightarrow \sum_{k=1}^{n} \frac{r(a_k)}{q'(a_k)} \exp(a_k t),$$

where the a_k's are the zeros of q.

Heaviside had no qualms about extending this result, as in the solution of the following problem (which is contained in Vol. 2 of his *Electromagnetic Theory*). Solve

$$\frac{\partial v}{\partial t}(x, t) = K \frac{\partial^2 v}{\partial x^2}(x, t) \quad (x \in (0, l), t > 0),$$

subject to the conditions

$$v(x, 0) = 0, \quad x \in (0, l),$$

$$v(0, t) = 1, \quad t > 0,$$

$$v(l, t) = 0, \quad t > 0.$$

If we write V for the Laplace transform of v, with respect to t, and write $q^2 = p/k$, the partial differential equation becomes, for each p,

$$\frac{d^2 V}{dx^2}(x, p) - q^2 V(x, p) = 0$$

subject to

$$V(0, p) = 1, \quad V(l, p) = 0.$$

The solution for V is thus

$$V(x, p) = \frac{\sinh(q(l - x))}{\sinh(ql)}.$$

The denominator is zero at $q = 0$ and $q = n\pi i/l$ (and so $p = -kn^2\pi^2/l^2$) and the "Expansion Theorem" (suddenly extended to an infinite expansion) gives

$$v = \frac{l - x}{l} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{l}\right) \exp\left(-\frac{kn^2\pi^2 t}{l}\right),$$

which happens to be the correct solution.

We can prove Heaviside's Theorem as follows.

Suppose that F has singularities at the points p_n and that these are all simple poles. Then the residue of $p \longrightarrow e^{pt}F(p)$ at p_n is

$$\lim_{p \to p_n} (p - p_n)e^{pt}F(p) = \exp(p_n t) \lim_{p \to p_n} (p - p_n)F(p),$$

and so the inverse Laplace transform of F is

$$\sum_{n=1}^{\infty} \exp(p_n t) \lim_{p \to p_n} (p - p_n)F(p),$$

provided F satisfies the conditions of Theorem 6.

In particular, if

$$F(p) = \frac{r(p)}{q(p)}$$

where r and q are each differentiable at each singularity, p_n, of F and $r(p_n) \neq 0$, then the residue at p_n of $p \longrightarrow F(p)e^{pt}$ is

$$\frac{r(p_n)}{q'(p_n)} \exp(p_n t), \quad \text{by Theorem 2 of } \textit{Unit 10},$$

and we can deduce *Heaviside's Expansion Theorem*: if all the singularities p_n, of F are simple poles and if F can be written as the quotient of two entire functions r/q then the inverse Laplace transform of F is

$$t \longrightarrow \sum_{n=1}^{\infty} \frac{r(p_n)}{q'(p_n)} \exp(p_n t),$$

provided F satisfies suitable conditions (for example, those of Theorem 6).

Summary

There are basically two types of inversion integral that we have dealt with: those involving a finite number of singularities and those with an infinite number. In each case, we have confined ourselves to isolated singularities. Cases where the plane has to be cut can be dealt with in a similar way (we proved Theorem 6 in a form suitable for cuts along the negative real axis), but the analysis is usually much more complicated. We shall not tackle such problems.

Self-Assessment Question

State the main steps in using the theory of residues to invert a Laplace transform.

Solution

See page 34.

14.5 PROBLEMS

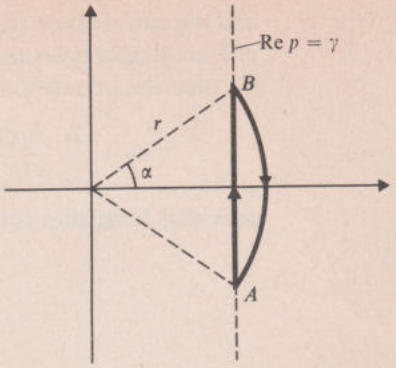


Fig. 12

1. Show directly that in Example 2 of Section 14.4, $f(t) = 0$ for $t < 1$.

   (Hint: Consider the closed contour Γ shown in Fig. 12, where the curved portion is an arc of a circle with centre the origin.)

2. Use the inversion integral to find $f(t)$ for each of the following specifications of $F(p)$. (Assume that the inversion integral is valid and can be represented as a sum of residues.)

   (i) $F(p) = \dfrac{1}{(p+2)(p-3)^2}$; (ii) $F(p) = \dfrac{1}{p^2+1}$;

   (iii) $F(p) = \dfrac{1}{(p^2+1)^2}$; (iv) $F(p) = \dfrac{1}{p\cosh p}$;

   (v) $F(p) = \dfrac{1}{p^2 \sinh p}$.

Solutions

1. By Cauchy's Theorem,

$$\int_\Gamma F(p)e^{pt}\,dt = 0.$$

So, if $\Gamma(r)$ denotes the curved part of Γ and AB the line segment from A to B we have

$$\int_{\Gamma(r)} F(p)e^{pt}\,dp = -\int_{AB} F(p)e^{pt}\,dp.$$

But $F(p) = pe^{-p}/(1+p^2)$, so

$$\left|\int_{\Gamma(r)} \frac{pe^{-p}e^{pt}}{1+p^2}\,dp\right| = \left|-\int_{-\alpha}^{\alpha} \frac{\left(re^{i\theta}\exp(re^{i\theta}(t-1)\right)}{1+r^2e^{2i\theta}} rie^{i\theta}\,d\theta\right|$$

$$\leqslant \int_{-\alpha}^{\alpha} \frac{r^2 e^{r\cos\theta(t-1)}}{|1+r^2e^{2i\theta}|}\,d\theta$$

$$\leqslant \int_{-\alpha}^{\alpha} e^{r\cos\theta(t-1)}\,d\theta,$$

which has limit 0 for large r, because $t < 1$ and $\cos\theta > 0$ (as $-\pi/2 < \theta < \pi/2$). Therefore

$$f(t) = \int_{\gamma-i\infty}^{\gamma+i\infty} F(p)e^{pt}\,dp = 0.$$

2. (i) The residue of $p \longrightarrow e^{pt}F(p)$ at -2 is $\dfrac{e^{-2t}}{25}$

   The residue of $p \longrightarrow e^{pt}F(p)$ at 3 is

$$\lim_{p\to 3}\left[\frac{d}{dz}\left(\frac{e^{pt}}{p+2}\right)\right] = \frac{te^{3t}}{5} - \frac{e^{3t}}{25}.$$

   Hence

$$f(t) = \frac{e^{-2t}}{25} + \frac{te^{3t}}{5} - \frac{e^{3t}}{25}.$$

   (ii) The residue of $p \longrightarrow e^{pt}F(p)$ at i is $\dfrac{e^{it}}{2i}$ and at $-i$ it is $\dfrac{e^{-it}}{-2i}$. Hence,

$$f(t) = \sin t$$

(iii) The residue of $p \longrightarrow e^{pt}F(p)$ at i is

$$\lim_{p\to i}\left[\frac{d}{dp}\left(\frac{e^{pt}}{(p+i)^2}\right)\right] = \lim_{p\to i}\left(\frac{.te^{pt}}{(p+i)^2} - \frac{2e^{pt}}{(p+i)^3}\right)$$

$$= \frac{te^{it}}{-4} + \frac{2e^{it}}{8i}.$$

Similarly, the residue at $-i$ is $\dfrac{te^{-it}}{-4} - \dfrac{2e^{-it}}{8i}$. Hence,

$$f(t) = \frac{\sin t - t\cos t}{2}.$$

(iv) Singularities occur at 0 and where $\cosh p = 0$, which gives $p = \dfrac{-i(2n-1)\pi}{2}$, where $n \in \mathbf{N}$. Each singularity is a simple pole. The residue of $p \longrightarrow e^{pt}F(p)$ at 0 is 1. The residue at $\dfrac{-i(2n-1)\pi}{2} = \alpha$ is

$$\frac{e^{\alpha t}}{\alpha \sinh \alpha} = \frac{2[\cos((2n-1)\pi t/2) - i\sin((2n-1)\pi t/2)]}{\pi(2n-1)\sin((2n-1)\pi/2)}$$

$$= \frac{(-1)^{n-1}2}{\pi(2n-1)}\cdot[\cos((2n-1)\pi t/2) - i\sin((2n-1)\pi t/2)].$$

Adding these residues over all (positive and negative) integer values of n, and taking account of the odd and even properties of sin and cos, we get

$$f(t) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos((2n-1)\pi t/2).$$

(v) Singularities are at 0 (pole of order three) and $n\pi i$ (simple poles), where $n \in \mathbf{N}$. The residue at 0 is the coefficient of $1/p$ in the Laurent expansion of $F(p)e^{pt}$. We have

$$\frac{e^{pt}}{p^2\sinh p} = \frac{e^{pt}}{p^2\left(p - \dfrac{p^3}{3!} + \cdots\right)}$$

$$= \frac{1}{p^3}\left(1 + pt + \frac{(pt)^2}{2} + \cdots\right)\left(1 + \frac{p^2}{3!} + \cdots\right).$$

The coefficient of $\dfrac{1}{p}$ is $\dfrac{t^2}{2} + \dfrac{1}{6}$.

The residue at $n\pi i$ is

$$\frac{e^{n\pi it}}{-(n\pi)^2\cosh n\pi i} = \frac{(-1)^{n+1}e^{n\pi it}}{n^2\pi^2}.$$

The sum of the residues is thus

$$\frac{t^2}{2} + \frac{1}{6} + \sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2}(\cos n\pi t + \sin n\pi t).$$

Thus

$$f(t) = \frac{t^2}{2} + \frac{1}{6} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi t.$$

14.6 USING CONTOUR INTEGRALS TO SOLVE DIFFERENTIAL EQUATIONS

We remarked earlier that one of the most important uses of the Laplace transform is in the solution of differential equations. The idea is to solve not for a function f but for the transform F. Then the inversion formula gives f as a contour integral:

$$f(t) = \frac{1}{2\pi i}\int_\Gamma e^{pt}F(p)\,dp,$$

where Γ is a suitably chosen contour. This suggests that perhaps we might be able to go directly to a solution in the form of a contour integral.

Example 1

As a very simple illustration we can take an equation which we know how to solve—a second order ordinary differential equation with constant coefficients:

$$af'' + bf' + cf = 0. \tag{1}$$

If we let $f(t) = \int_\Gamma e^{pt}F(p)\,dp$ and assume that we can differentiate under the integral sign to give

$$f'(t) = \int_\Gamma pe^{pt}F(p)\,dp \quad \text{and} \quad f''(t) = \int_\Gamma p^2e^{pt}F(p)\,dp,$$

then direct substitution in (1) gives

$$\int_\Gamma (ap^2 + bp + c)e^{pt}F(p)\,dp = 0,$$

where the function F and the contour Γ have to be suitably chosen.

Since we wish to find a non-zero solution of (1), F and Γ must be chosen so that F has poles inside Γ, for then the integral for $f(t)$ is not zero.

Suppose the polynomial $ap^2 + bp + c$ has zeros at α and β. If we assume that

$$F(p) = \frac{A}{p-\alpha} + \frac{B}{p-\beta}, \quad \text{where } A \text{ and } B \text{ are constants,}$$

then F has simple poles at α and β and hence so has $p \longrightarrow e^{pt}F(p)$; but $p \longrightarrow (ap^2 + bp + c)e^{pt}F(p)$ has only removable singularities (by Theorem 1 of *Unit 8*). So with this choice for $F(p)$,

$$f(t) = \int_\Gamma e^{pt}F(p)\,dp$$

is a non-zero solution when Γ encloses either or both of α and β. Fig. 13 shows one suitable contour.

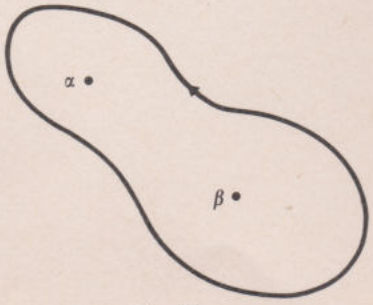


Fig. 13

The residues at α and β are simply $Ce^{\alpha t}$ and $De^{\beta t}$ respectively (where C and D are constant) and so

$$f(t) = Ce^{\alpha t} + De^{\beta t}$$

is the solution, as we knew all along. The case $\alpha = \beta$ is the subject of the self-assessment question on page 43.

Example 2

The equation

$$(a_0 + a_1 t)f''(t) + (b_0 + b_1 t)f'(t) + (c_0 + c_1 t)f(t) = 0,$$

with linear coefficients can also be solved by looking for a solution of the form

$$f(t) = \int_\Gamma e^{pt} F(p)\, dp.$$

This time, direct substitution gives

$$\int_\Gamma [(a_0 p^2 + b_0 p + c_0) + (a_1 p^2 + b_1 p + c_1)t] e^{pt} F(p)\, dp = 0, \tag{2}$$

where the function F and contour Γ have to be suitably chosen. The integral in (2) would be considerably simplified if we could find a function g (which depends on F, the a_i's, b_i's and c_i's) such that the integral reduces to

$$\int_\Gamma \frac{d}{dp}\big(e^{pt} g(p)\big)\, dp = 0,$$

for then we have simply that $e^{pt} g(p)|_\Gamma = 0$.

If such a function g could be found, we would have

$$\int_\Gamma \big(g'(p) + t g(p)\big) e^{pt}\, dp = 0,$$

and comparing this with the integral involving F, Equation (2), we see that g has to be chosen so that

$$g'(p) = (a_0 p^2 + b_0 p + c_0)F(p) \tag{3}$$

and

$$g(p) = (a_1 p^2 + b_1 p + c_1)F(p). \tag{4}$$

(Note the Equation (4) enables us to obtain F once we have found g.) From (3) and (4), we require

$$\frac{g'(p)}{g(p)} = \frac{a_0 p^2 + b_0 p + c_0}{a_1 p^2 + b_1 p + c_1},$$

which expresses g in terms of quantities originally given in the problem, by-passing the function F. If $p \longrightarrow a_1 p^2 + b_1 p + c$ has zeros at α and β ($\alpha \neq \beta$) then we can find constants A, B and C such that

$$\frac{g'(p)}{g(p)} = A + \frac{B}{p - \alpha} + \frac{C}{p - \beta};$$

we can integrate this to give

$$L\big(g(p)\big) = Ap + BL(p - \alpha) + CL(p - \beta),$$

where L is some branch of the logarithm, and so

$$g(p) = e^{Ap}(p - \alpha)^B (p - \beta)^C. \tag{5}$$

Comparing (4) and (5), we see that

$$F(p) = \frac{1}{a_1} e^{Ap}(p - \alpha)^{B-1}(p - \beta)^{C-1},$$

and so,

$$f(t) = \int_\Gamma e^{(t+A)p}(p - \alpha)^{B-1}(p - \beta)^{C-1}\, dp.$$

The contour Γ is chosen so that $e^{pt}g(p)|_\Gamma = 0$, that is

$$e^{(t+A)p}(p-\alpha)^B(p-\beta)^C|_\Gamma = 0,$$

taking care to ensure that the integral for $f(t)$ is not zero as well!

Example 3

Airy's equation,

$$f''(t) - tf(t) = 0,$$

is a special case of the general equation considered in Example 2. Following the method of that example, we look for solutions of the form

$$f(t) = \int_\Gamma e^{pt}F(p)\,dp.$$

Direct substitution gives

$$\int_\Gamma e^{pt}(p^2 - t)F(p)\,dp = 0,$$

and if this is to be compared with

$$\int_\Gamma \frac{d}{dp}\left(e^{pt}g(p)\right)dp = 0,$$

we must choose g so that

$$g'(p) = p^2F(p) \quad \text{and} \quad g(p) = -F(p).$$

Thus

$$\frac{g'(p)}{g(p)} = -p^2,$$

from which

$$g(p) = K\exp(-p^3/3), \quad \text{where } K \text{ is a constant,}$$

and so

$$F(p) = -K\exp(-p^3/3).$$

So with a suitably chosen contour Γ, and ignoring $-K$, we see that

$$f(t) = \int_\Gamma \exp(tp - p^3/3)\,dp$$

is a solution. By "suitably chosen", we mean that

$$\exp(tp - p^3/3)|_\Gamma = 0,$$

and also that $f(t)$ is non-zero; but, since $p \longrightarrow \exp(tp - p^3/3)$ is an entire function, any *closed* contour that would ensure that $\exp(tp - p^3/3)|_\Gamma = 0$ would also make $f(t)$ zero. So no closed contour will serve our purpose, and the question raised is whether *any* suitable contour exists. For any non-closed contour we shall not have $\exp(tp - p^3/3)|_\Gamma = 0$, because $\exp(tp - p^3/3) \neq 0$ for any p. To get over this problem we introduce the idea of a "generalized contour" which is (unlike an ordinary contour) unbounded, and try to choose Γ to be such a contour that "goes to infinity" in directions chosen so that $\exp(tp - p^3/3)$ approaches zero as p gets large along Γ.

If we let $p = re^{i\theta}$ (usually a good ploy when an exponential is involved), we have

$$|\exp(tp - p^3/3)| = |\exp(tr\cos\theta - r^3\cos 3\theta/3|.$$

This will approach zero as r increases if

$$\cos 3\theta > 0.$$

Thus θ will have to be chosen so that

$$-\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} < 3\theta < \frac{5\pi}{2} \quad \text{or} \quad \frac{-5\pi}{2} < 3\theta < \frac{-3\pi}{2},$$

that is

$$-\frac{\pi}{6} < \theta < \frac{\pi}{6} \quad \text{or} \quad \frac{\pi}{2} < \theta < \frac{5\pi}{6} \quad \text{or} \quad \frac{-5\pi}{6} < \theta < \frac{-\pi}{2}.$$

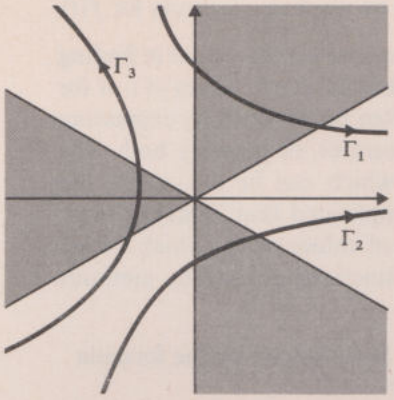


Fig. 14

Thus "contours" such as $\Gamma_1, \Gamma_2, \Gamma_3$ in Fig. 14 that go to infinity in the unshaded regions will give solutions. We shall return to integrals of this type later (page 60).

Summary

In this short section we have demonstrated the direct use of contour integrals in solving ordinary differential equations, and we have seen the need in certain cases to introduce the idea of a generalized contour.

Self-Assessment Question

Complete Example 1 for the case $\alpha = \beta$.

Solution

If $\alpha = \beta$, then $F(p) = \dfrac{A}{(p-\alpha)^2}$, and $p \longrightarrow e^{pt}F(p)$ has a pole of order 2 at α and the residue there is $te^{\alpha t}$; thus $f(t) = te^{\alpha t}$ is a solution, as well as $e^{\alpha t}$.

14.7 ASYMPTOTIC EXPANSIONS

Whether we try to solve a differential equation for a function f using the Laplace transform or by a direct contour integral substitution, we end up with an integral for $f(t)$ to evaluate in the complex plane. This integral involves a parameter (we have used t) that is the variable in the original problem. As you might expect, real-life situations are not usually so kind as to present us with the sort of integrals that appear in examination papers. Usually it is not possible to evaluate these integrals in the sense of finding an algebraic formula for $f(t)$.

It is fortunate, then, that in many practical problems we are interested in finding out about $f(t)$ for particular values of t – very often for small values of t or for large values of t. The required information can often be obtained by expressing the integrand in a different form, changing the contour, or possibly both. The idea can be usefully applied also to integrals which *can* be evaluated; the evaluation may be so tricky and technically complicated that it makes more sense to consider its behaviour only in the range of values of t in which we are interested. (The same principle applies also to functions defined by methods other than integrals.)

For example, the *factorial function* $n \longrightarrow n!$, $n \in \mathbf{N}$, is defined by the formula

$$n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1.$$

But there is a result due to Stirling* which says that

$$\lim_{n\to\infty} \frac{n!}{e^{-n}n^n\sqrt{2\pi n}} = 1:$$

in other words, for large values of n, the value of $n!$ is approximately given by

$$e^{-n}n^n\sqrt{2\pi n},$$

Furthermore, the approximation gets better and better as n increases. This is a typical example of an *asymptotic approximation.*

Another example, is one which you will meet if you tackle *Unit 15, Number Theory*. (See page 6 of that unit.) If $\pi(x)$ is the number of prime numbers not exceeding x, then

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1.$$

In other words, for large values of x, $\pi(x)$ is given approximately by $\dfrac{x}{\log x}$.

Our third example is closer to the type of problem we shall investigate here.

Example 1

Find an approximation to $f(t) = \displaystyle\int_1^t \frac{e^x}{x}dx$ which is "good" for large values of t.

Integration by parts gives

$$f(t) = \frac{e^x}{x}\bigg|_1^t + \int_1^t \frac{e^x}{x^2}dx,$$

and another integration by parts gives

$$f(t) = \frac{e^x}{x}\bigg|_1^t + \frac{e^x}{x^2}\bigg|_1^t + 2\int_1^t \frac{e^x}{x^3}dx.$$

* *James Stirling* (1692–1770) an English mathematician of the Newtonian school, obtained this result in 1730.

Continuing in this way, we obtain

$$f(t) = e^x\left(\frac{1}{x} + \frac{1}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots + \frac{(n-1)!}{x^n}\right)\Bigg|_1^t + n!\int_1^t \frac{e^x}{x^{n+1}}\,dx.$$

Thus

$$e^{-t}f(t) \simeq \frac{1}{t} + \frac{1}{t^2} + \frac{2!}{t^3} + \cdots + \frac{(n-1)!}{t^n}$$

with an error term given by

$$-e^{1-t}(1 + 1 + 2! + 3! + \cdots + (n-1)!) + e^{-t}n!\int_1^t \frac{e^x}{x^{n+1}}\,dx.$$

It seems reasonable that for large t the error term is small, and hence it would seem that $e^{-t}f(t)$ is given approximately by

$$\frac{1}{t} + \frac{1}{t^2} + \frac{2!}{t^3} + \frac{3!}{t^4} + \cdots + \frac{(n-1)!}{t^n}$$

for large values of t. The series $\sum_{n=1}^{\infty} \frac{(n-1)!}{t^n}$ does *not* converge for any value of t but, as we shall soon see, by taking t large enough, any given number of terms of the series can be made to approximate as close as we please to $e^{-t}f(t)$. The expression

$$\frac{1}{t} + \frac{1}{t^2} + \frac{2!}{t^3} + \cdots + \frac{(n-1)!}{t^n} + \cdots$$

is called an *asymptotic expansion* of $t \longrightarrow e^{-t}f(t)$.

The essential features of the expansion in the previous example that justify the term "asymptotic expansion" can be expressed in more general terms as follows.

Definition

Let $S_n(t) = a_0 + \frac{a_1}{t} + \cdots + \frac{a_n}{t^n}$, $n \in \mathbf{N}$. Then, if *for every fixed n*

$$\lim_{t\to\infty} t^n(f(t) - S_n(t)) = 0,$$

we say that $a_0 + \frac{a_1}{t} + \cdots + \frac{a_n}{t^n} + \cdots$ is an **asymptotic expansion** of f, for large t, and we express this in symbols by writing

$$f(t) \sim S_n(t),$$

which is read as "$f(t)$ is asymptotic to $S_n(t)$".

We also write

$$f(t) \sim g(t)\left(a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n}\right)$$

to mean that

$$\lim_{t\to\infty} t^n(f(t) - g(t)S_n(t)) = 0$$

or

$$\frac{f(t)}{g(t)} \sim a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n}, \quad g(t) \neq 0.$$

You should make sure that you are clear about the difference between asymptotic expansions and convergent series. If the sequence $\{S_n(t)\}$ *converges* to $f(t)$,

then we mean that for certain values of t we can get as close as we like to $f(t)$ by taking n sufficiently large. If $f(t) \sim S_n(t)$, for large t, then we mean that for *every fixed* value of n, the difference between $S_n(t)$ and $f(t)$ can be made as small as we please by *making t large enough*. In general, an asymptotic expansion will not be a convergent series.

To emphasize even further the difference between asymptotic expansions and convergent series, look again at the expansion we obtained in Example 1:

$$e^{-t}\int_1^t \frac{e^x}{x}\,dx \sim \frac{1}{t} + \frac{1}{t^2} + \frac{2!}{t^3} + \cdots + \frac{(n-1)!}{t^n},$$

and consider a specific value of t, for example $t = 10$. How do we choose n in order to approximate $e^{-t}\int_1^{10} \frac{e^x}{x}\,dx$? Of course, the answer depends on the accuracy required, but by inspection of the terms of the approximation it is easy to see that terms *after* the eleventh, $\frac{10!}{10^{11}}$ (corresponding to $t = 10, n = 11$), increase as n increases and so there is no point in including terms after this one in *any* approximation for $t = 10$. On the other hand, if we are interested in an approximation for a larger value of t than 10, then the approximation given by the first eleven terms will be better for this case than it is for the case $t = 10$. Also, more terms may be taken in the approximation, before the approximation begins to deteriorate. For example, with $t = 12$, there is no point in including terms after the thirteenth in *any* approximation.

In the next example, we show that the result of Example 1 is indeed an asymptotic expansion.

Example 2

From Example 1, we have

$$t^n\left[e^{-t}f(t) - \left(\frac{1}{t} + \frac{1}{t^2} + \frac{2!}{t^3} + \cdots + \frac{(n-1)!}{t^n}\right)\right] = R_n,$$

where $R_n = -e^{1-t}t^n\big(1 + 1 + 2! + 3! + \cdots + (n-1)!\big) + n!e^{-t}t^n\int_1^t \frac{e^x}{x^{n+1}}\,dx.$

The first group of terms has limit zero for large t. (See Theorem 17–6 in **Spivak**.) To deal with the integral term, we can integrate by parts to give

$$n!e^{-t}t^n\frac{e^x}{x^{n+1}}\bigg|_1^t + (n+1)!e^{-t}t^n\int_1^t \frac{e^x}{x^{n+2}}\,dx. \tag{1}$$

Consider now this final integral:

$$\begin{aligned}\int_1^t \frac{e^x}{x^{n+2}}\,dx &= \int_1^{t/2} \frac{e^x}{x^{n+2}}\,dx + \int_{t/2}^t \frac{e^x}{x^{n+2}}\,dx\\ &\leqslant e^{t/2}\int_1^{t/2} \frac{1}{x^{n+2}}\,dx + e^t\int_{t/2}^t \frac{1}{x^{n+2}}\,dx\\ &= -\frac{e^{t/2}}{n+1}\left(\frac{2^{n+1}}{t^{n+1}} - 1\right) - \frac{e^t}{n+1}\left(\frac{1}{t^{n+1}} - \frac{2^{n+1}}{t^{n+1}}\right).\end{aligned} \tag{2}$$

It can now be seen from (1) and (2) and Theorem 17–6 of **Spivak** that the integral term in R_n has limit zero for large t. This completes the proof that the result of Example 1 is indeed an asymptotic expansion.

Example 3

In the expansion dealt with in Examples 1 and 2 if we take $n = 3$ and $t = 10$, the error term turns out to be less than 0.001. In other words, for $t = 10$, three terms of the asymptotic expansion give the integral correct to two decimal places.

Asymptotic Expansions for Laplace Transforms

A normal treatment of asymptotic expansions follows the usual procedure; we show that any given function has only one asymptotic expansion, discover how to add asymptotic expansions, multiply them, integrate them, and so on. We shall not do this, because our purpose is merely to show how complex variables come into the calculation of asymptotic expansions.

The first result that we look at has particular reference to the earlier sections of this unit; it tells us how to compute the asymptotic expansion of a Laplace transform.

Theorem 7 (Watson's Lemma)

Given that $f(t) = \sum_{k=0}^{\infty} a_k t^k$, $|t| < r$, and that for some b such that $0 < b < r$ constants K (>0) and α can be found such that

$$|f(t)| < Ke^{\alpha t} \quad \text{for} \quad |t| \geqslant b,$$

then, the Laplace transform F of f has asymptotic expansion given by

$$F(p) = \int_0^{\infty} e^{-pt} f(t)\, dt \sim \sum_{k=0}^{n} \frac{a_k k!}{p^{k+1}},$$

for p large and real.

Remark Watson's Lemma tells us that we can find an asymptotic expansion for the Laplace transform of a function f by expanding f in a Taylor series about 0 and taking the Laplace transform term by term.

Proof

We may write

$$F(p) = \sum_{k=0}^{n} \int_0^{\infty} e^{-pt} a_k t^k\, dt + R_n$$

$$= \sum_{k=0}^{n} \frac{a_k k!}{p^{k+1}} + R_n, \quad \text{since } \mathcal{L}[t \longrightarrow t^k] = p \longrightarrow k!/p^{k+1}.$$

We have to show that $\lim_{p \to \infty} p^n R_n = 0$, for every fixed n. We have

$$|R_n| = \left| F(p) - \sum_{k=0}^{n} \frac{a_k k!}{p^{k+1}} \right|$$

$$= \left| \int_0^{\infty} e^{-pt} \left(f(t) - \sum_{k=0}^{n} a_k t^k \right) dt. \right|$$

We now try to get an upper estimate for the integrand.

Consider $f(t) - \sum_{k=0}^{n} a_k t^k$ for $|t| < b$; we have

$$\left| f(t) - \sum_{k=0}^{n} a_k t^k \right| = |t^n (a_{n+1} t + a_{n+2} t^2 + \cdots)|.$$

The series $\sum_{k=n+1}^{\infty} a_k t^k$ is absolutely convergent and so

$$\left| \sum_{k=n+1}^{\infty} a_k t^{k-n} \right| \leqslant \sum_{k=n+1}^{\infty} |a_k|\, |t^{k-n}| \leqslant \sum_{k=n+1}^{\infty} |a_k| b^k = A, \text{ say.}$$

Thus, for $t < b$,

$$\left| f(t) - \sum_{k=0}^{n} a_k t^k \right| \leqslant A|t^n|.$$

Also, for $t \geqslant b$,

$$\left| f(t) - \sum_{k=0}^{n} a_k t^k \right| \leqslant Be^{\alpha t} \text{ for some number } B.$$

Thus for all $t > 0$,

$$\left| f(t) - \sum_{k=0}^{n} a_k t^k \right| \leqslant Ce^{\alpha t}|t^n| \text{ for some number } C,$$

and so

$$|R_n| \leqslant \int_0^\infty e^{-pt} Ce^{\alpha t}|t^n|\, dt$$

$$= \frac{C \cdot n!}{(p-\alpha)^{n+1}}, \quad p > \alpha.$$

Thus for every fixed n,

$$|p^n R_n| < \frac{C \cdot n! p^n}{(p-\alpha)^{n+1}},$$

which has limit zero for large p. ■

This completes the proof of the theorem as stated. Previously, we have found it useful to consider the Laplace transform $p \longrightarrow F(p)$ for complex values of p. Watson's Lemma can be extended to such values, the proof needing modification only in the final stages. This version of the lemma reveals a characteristic of asymptotic expansions in a complex variable—they are valid for p in some sector of the complex plane. For the statement of Watson's Lemma in this form and the proof, see Levinson, N. and Redheffer, R. M. *Complex Variables* Holden–Day, 1970.

Example 4

Find an asymptotic expansion for $\int_x^\infty e^{-t}t^7\, dt$, where x is real and positive.

To get the form of integral in Watson's Lemma, we substitute $t = u + x$, so as to get the limits of integration right. This gives

$$\int_0^\infty e^{-(u+x)}(u+x)^7\, du = e^{-x}x^7 \int_0^\infty e^{-u}\left(1 + \frac{u}{x}\right)^7 du.$$

Now let $u = xt$, so as to get the integrand in the right form. This gives

$$e^{-x}x^8 \int_0^\infty e^{-xt}(1+t)^7\, dt.$$

To apply Watson's Lemma, take b in the lemma to be $\frac{1}{2}$, say. For $|t| < \frac{1}{2}$, $(1+t)^7$ can be expanded in a (finite) Taylor series and for $|t| \geqslant \frac{1}{2}$ we can find a number α such that $|(1+t)^7|$ is bounded by $e^{\alpha t}$. Thus Watson's Lemma applies and we get

$$\int_x^\infty e^{-t}t^7\, dt \sim e^{-x}x^8 \int_0^\infty e^{-xt}\left(1 + 7t + \frac{7 \cdot 6}{2}t^2 + \cdots + t^7\right) dt.$$

Thus

$$\int_x^\infty e^{-t}t^7\, dt \sim e^{-x}x^8\left(\frac{1}{x} + \frac{7}{x^2} + \frac{7 \cdot 6}{x^3} + \cdots + \frac{7!}{x^8}\right).$$

(Notice that these integrations are easily performed once you recognize that each integral has the form of a Laplace transform—and the Laplace transform of $t \longrightarrow t^k$ is $p \longrightarrow \dfrac{k!}{p^{k+1}}$. Note also that since the function concerned has a *finite* Taylor series, the asymptotic expansion has $a_n = 0$ for $n \geqslant 9$.)

Summary

We have discussed the nature of an asymptotic approximation and given a definition of an asymptotic expansion. We have also proved a theorem (Watson's Lemma) which enables us to calculate the asymptotic expansion of an integral having the form of a Laplace transform.

Self-Assessment Questions

1. Define the term "asymptotic expansion" of a function f, and explain the notation $f(t) \sim g(t)$.
2. Classify the following statements as True or False.
   (i) If $f(t) \sim S_n(t)$, for large t, then given $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that
   $$|S_n(t) - f(t)| < \varepsilon \text{ for } n > N.$$
   (ii) If $f(t) \sim S_n(t)$, for large t, then for any fixed n, given $\varepsilon > 0$, there exists $T \in \mathbf{R}$ such that
   $$|S_n(t) - f(t)| < \varepsilon \text{ for } t > T.$$
   (iii) If $a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n} \sim f(t)$, then the series $a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots$ necessarily converges.
3. Describe briefly the implication of Watson's Lemma for Laplace transforms.

Solutions

1. See page 45.
2. (i) False. (ii) True. (iii) False.
3. See the Remark on page 47.

14.8 PROBLEMS

1. Find an asymptotic expansion for $\int_x^\infty e^{-t}t^n\,dt$, where n is an integer and x is real and positive. (Hint: See Example 4 of Section 14.7.)

2. Show by example that two different functions can have the same asymptotic expansion. (Hint: Recall that $\lim_{t\to\infty} t^n e^{-t} = 0$.)

3. Let

$$f(t) \sim a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n}$$

and

$$f(t) \sim b_0 + \frac{b_1}{t} + \frac{b_2}{t^2} + \cdots + \frac{b_n}{t^n}.$$

Use mathematical induction to show that $a_k = b_k, k = 0, 1, 2, \ldots$. (This means that an asymptotic expansion for a function is unique.)

4. Given that

$$f(t) \sim a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_n}{t^n}$$

and

$$g(t) \sim b_0 + \frac{b_1}{t} + \frac{b_2}{t^2} + \cdots + \frac{b_n}{t^n},$$

show that

$$f(t) + g(t) \sim (a_0 + b_0) + \frac{(a_1 + b_1)}{t} + \cdots + \frac{(a_n + b_n)}{t^n},$$

and

$$f(t)\cdot g(t) \sim a_0b_0 + \frac{a_0b_1 + a_1b_0}{t} + \cdots + \frac{a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0}{t^n}.$$

5. (a) Assuming that

$$\int_0^\infty \frac{e^{-pt}}{\sqrt{1+t}}\,dt \sim \frac{1}{p}\left(1 - \frac{1}{2p} + \frac{1\cdot 3}{(2p)^2} - \frac{1\cdot 3\cdot 5}{(2p)^3} + \cdots + \frac{(-1)^n 1\cdot 3\cdot \ldots \cdot (2n-1)}{(2p)^n}\right)$$

is valid for $p = ix, x > 0$, find the asymptotic expansion for

$$\int_0^\infty \frac{e^{-ixt}}{\sqrt{1+t}}\,dt$$

and deduce that

$$\int_x^\infty \frac{\sin(u-x)}{\sqrt{u}}\,du = \frac{a(x)}{\sqrt{x}},$$

where $a(x) \sim 1 - \frac{1\cdot 3}{(2x)^2} + \frac{1\cdot 3\cdot 5\cdot 7}{(2x)^4} - \cdots + \frac{(-1)^n 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{(2x)^{2n}}$,

and

$$\int_x^\infty \frac{\cos(u-x)}{\sqrt{u}}\,du = \frac{b(x)}{\sqrt{x}},$$

where $b(x) \sim \dfrac{1}{2x} - \dfrac{1\cdot 3\cdot 5}{(2x)^3} + \cdots + \dfrac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)}{(2x)^{2n-1}}$.

(b) *If you have time*, deduce that

$$\int_x^\infty \frac{\sin u}{\sqrt{u}}\,du = \frac{1}{\sqrt{x}}(a(x)\cos x + b(x)\sin x)$$

and

$$\int_x^\infty \frac{\cos u}{\sqrt{u}}\,du = \frac{1}{\sqrt{x}}(b(x)\cos x - a(x)\sin x).$$

(These integrals are called *Fresnel's integrals.*)

6. Show that

$$\int_0^\infty e^{-x\sinh u}\,du = \int_0^\infty \frac{e^{-xt}}{\sqrt{1+t^2}}\,dt$$
$$\sim \frac{1}{x}\left(1 - \frac{1}{x^2} + \frac{1^2\cdot 3^2}{x^4} - \frac{1^2\cdot 3^2\cdot 5^2}{x^6} + \cdots + \frac{(-1)^n\cdot 1^2\cdot 3^2\cdot\ldots\cdot(2n-1)^2}{x^{2n}}\right), \quad x > 0.$$

(Hint: Use Watson's Lemma and the fact that the Laplace transform of $t \longrightarrow t^n$ is $p \longrightarrow \dfrac{n!}{p^{n+1}}$.)

Solutions

1. Using the same method as in Example 4 of Section 14.7, we obtain

$$\int_x^\infty e^{-t}t^n\,dt \sim e^{-x}x^{n+1}\left(\frac{1}{x} + \frac{n}{x^2} + \frac{n(n-1)}{x^3} + \cdots + \frac{1}{x^n}\right).$$

(The integral $\displaystyle\int_x^\infty e^{-t}t^n\,dt$ is of considerable interest in its own right by virtue of its similarity to the factorial function $n \longrightarrow n!$, that is

$$n \longrightarrow \int_0^\infty e^{-t}t^n\,dt.$$

The integral $\displaystyle\int_x^\infty e^{-t}t^n\,dt$ is related to the factorial function because

$$\int_0^x e^{-t}t^n\,dt + \int_x^\infty e^{-t}t^n\,dt = \int_0^\infty e^{-t}t^n\,dt = \Gamma(n+1) = n!.$$

The function $n \longrightarrow \displaystyle\int_0^x e^{-t}t^n\,dt$ is called the *incomplete factorial function* and the function $n \longrightarrow \displaystyle\int_x^\infty e^{-t}t^n\,dt$ is the *complementary incomplete factorial function.*)

2. The functions $f(t)$ and $f(t) + e^{-t}$ have the same asymptotic expansions because $\lim_{t\to\infty} t^n e^{-t} = 0$ for any positive integer value of n.

3. We have $\lim_{t\to\infty}[f(t) - a_0] = 0$, and so $a_0 = \lim_{t\to\infty} f(t)$; similarly, $b_0 = \lim_{t\to\infty} f(t)$. So $a_0 = b_0$.

Suppose $a_k = b_k$, $k = 0, 1, 2, \ldots, n$; then given $\varepsilon > 0$, there is a T such that, for $t > T$.

$$\left| t^{n+1}\left[f(t) - \left(a_0 + \frac{a_1}{t} + \cdots + \frac{a_{n+1}}{t^{n+1}} \right) \right] \right| < \varepsilon$$

and

$$\left| t^{n+1}\left[f(t) - \left(b_0 + \frac{b_1}{t} + \cdots + \frac{b_{n+1}}{t^{n+1}} \right) \right] \right| < \varepsilon.$$

Now

$$\begin{aligned} |a_{n+1} - b_{n+1}| &= |t^{n+1}| \left| \frac{a_{n+1} - b_{n+1}}{t^{n+1}} \right| \\ &= |t^{n+1}| \left| \left[f(t) - \left(b_0 + \frac{b_1}{t} + \cdots + \frac{b_{n+1}}{t^{n+1}} \right) \right] \right. \\ &\qquad \left. - \left[f(t) - \left(a_0 + \frac{a_1}{t} + \cdots + \frac{a_{n+1}}{t^{n+1}} \right) \right] \right| \\ &< \varepsilon + \varepsilon, \quad \text{by the triangle inequality.} \end{aligned}$$

Thus $|a_{n+1} - b_{n+1}| < 2\varepsilon$ for any positive ε, and so $a_{n+1} = b_{n+1}$.

4. Let

$$S_n(t) = a_0 + \frac{a_1}{t} + \cdots + \frac{a_n}{t}$$

and

$$T_n(t) = b_0 + \frac{b_1}{t} + \cdots + \frac{b_n}{t}, \quad \text{where } n \in \mathbf{N}.$$

We have to show that for every fixed n

$$\lim_{t \to \infty} t^n \big(f(t) + g(t) - S_n(t) - T_n(t) \big) = 0.$$

Since $\lim_{t \to \infty} t^n \big(f(t) - S_n(t) \big) = 0$ and $\lim_{t \to \infty} t^n \big(g(t) - T_n(t) \big) = 0$ for every fixed n, the result follows.

For the product we have to show that for every fixed n

$$\lim_{t \to \infty} t^n \left(f(t) \cdot g(t) - \sum_{k=0}^{n} c_k / t^k \right) = 0,$$

where $c_k = \sum_{i=0}^{k} a_i b_{i-k}$.

As usual we add in some terms to give something that we can deal with. Consider

$$\lim_{t \to \infty} t^n [\big(f(t) - S_n(t) \big) g(t) + S_n(t) \big(g(t) - T_n(t) \big) + \big(S_n(t) T_n(t) - \sum_{k=0}^{n} c_k / t^k \big)] \qquad (1)$$

Certainly $\lim_{t \to \infty} t^n \big(f(t) - S_n(t) \big) = 0$, $\lim_{t \to \infty} g(t) = b_0$, $\lim_{t \to \infty} S_n(t) = a_0$, $\lim_{t \to \infty} t^n \big(S(t) - T_n(t) \big) = 0$.

Also, since $S_n(t) T_n(t) - \sum_{k=0}^{n} c_k / t^k$ consists of a sum of terms from one in $1/t^{n+1}$ to one in $1/t^{2n}$, it follows that

$$\lim_{t \to \infty} t^n \left(S_n(t) T_n(t) - \sum_{k=0}^{\infty} c_k / t^k \right) = 0.$$

Since the above results hold for every fixed n, the limit in (1) is zero and the result follows.

5. (a) $\displaystyle\int_0^\infty \frac{e^{-ixt}}{\sqrt{1+t}} dt \sim \frac{-i}{x} \left(1 + \frac{i}{2x} - \frac{1 \cdot 3}{(2x)^2} - \frac{i \cdot 1 \cdot 3 \cdot 5}{(2x^3)} + \cdots + \frac{(-i)^n 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{(2x)^n} \right).$

Substituting $t = \dfrac{u - x}{x}$, we obtain

$$\begin{aligned} \int_x^\infty \frac{e^{-i(u-x)}}{\sqrt{u}\sqrt{x}} du \sim \frac{-i}{x} \left(1 + \frac{i}{2x} - \frac{1 \cdot 3}{(2x)^2} - \frac{i \cdot 1 \cdot 3 \cdot 5}{(2x)^3} + \cdots \right. \\ \left. + \frac{(-i)^n 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{(2x)^n} \right) \end{aligned}$$

The results follow by expressing $e^{-i(u-x)}$ in trigonometric form and equating real and imaginary parts.

(b) The final results follow from the identities

$$\sin(u - x) = \sin u \cos x - \cos u \sin x$$

and

$$\cos(u - x) = \cos u \cos x + \sin u \sin x,$$

by taking the $\sin x$ and $\cos x$ factors out of the integrals and thus obtaining two equations for Fresnel's integrals.

6. The relationship between $\int_0^\infty e^{-xt}/\sqrt{(1 + t^2)}\, dt$ and $\int_0^\infty e^{-x \sinh u}\, du$ is established by substituting $t = \sinh u$.

$$\int_0^\infty \frac{e^{-xt}}{\sqrt{1 + t^2}} dt = \int_0^\infty e^{-xt}[1 - \tfrac{1}{2}t^2 + (-\tfrac{1}{2})\cdot(-\tfrac{3}{2})t^4 + (-\tfrac{1}{2})\cdot(-\tfrac{3}{2})\cdot(-\tfrac{5}{2})t^5 + \cdots]\, dt$$

$$\sim \frac{1}{x} - \frac{1}{2}\cdot\frac{2}{x^3} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{4!}{x^5} + \cdots + \frac{a_n}{x^{2n+1}},$$

where $a_n = \dfrac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)(2n)!}{2^n x^{2n+1}}$, $\quad n \in \mathbf{N}$.

The 2^n in the denominator will provide enough factors to turn each of the even numbers in $(2n)!$ into odd numbers giving

$$a_n = \frac{(-1)^n 1^2 \cdot 3^2 \cdot 5^2 \cdot \ldots \cdot (2n - 1)^2}{x^{2n+1}}.$$

14.9 THE METHOD OF STEEPEST DESCENT

As we saw in Example 4 of Section 14.8, Watson's Lemma can often be applied to *real* integrals that appear on the face of things to be unlike the form required by the lemma. The required form is obtained by transforming the integrand in some way. With integrals in the *complex* plane, we have far more flexibility for we are also at liberty to transform the path of integration provided that we do not cross any singularities of the integrand in the process.

It was fairly clear in Example 4 of Section 14.8 what sort of transformation of the integrand would help; we were aiming for an integral of the form $\int_0^\infty e^{-pt} f(t)\,dt$ because we knew how to find an asymptotic expansion* for such an integral. But how can a change of contour help us? To answer this question let us think for a moment just what we are trying to do when finding an asymptotic expansion for an integral. Essentially, we are finding an approximation to an integral,

$$g(p) = \int_\Gamma \phi(z)\,dz$$

for example, where the function ϕ or the contour Γ involves a parameter p.

If Γ is of finite length L, and if M is the maximum value of $|\phi(z)|$ on Γ then we know, from the Estimation Theorem, that

$$|g(p)| \leqslant LM.$$

(Note that L and M depend on Γ.)

We are often faced with integrals for which $\phi(z)$ changes rather violently for small changes in z. This has two effects. First, it means that the major contribution to $g(p)$ may come only from a small portion of the contour.

Secondly, a small deformation of the contour Γ may result in a large change in M, the maximum value of $|\phi(z)|$ on Γ, but almost no change in L, the length of Γ. We might hope therefore to reduce the upper estimate LM by choosing a contour Γ for which M is as small as possible.

There may be many such contours and so we try to choose a contour Γ on which $|\phi(z)|$ is close to its maximum value on Γ only on a relatively small part of Γ; in other words, we choose Γ such that $|\phi(z)|$ decreases as quickly as possible as z moves along Γ away from the value at which the maximum occurs. It may then be possible to parametrize Γ, obtain a real integral and apply a method suitable for real integrals, such as Watson's Lemma.

To see how to put this idea into practice we consider the problem geometrically. We are interested in the behaviour of $|\phi(z)|$ and this can be visualized as a surface, namely the graph of $(x, y) \longrightarrow w(x, y)$, where

$$w(x, y) = |\phi(x + iy)|.$$

(We shall refer to this graph as the "surface representing ϕ".)

We are looking for a contour Γ with fixed ends, P and Q say, such that the maximum value of w on Γ is as small as possible. We can visualize the problem as that of finding a route through a mountain range from P(given) to Q(given), such that the highest point we have to climb to is as low as possible and that we stay as low as possible for as long as possible.

* From here on when we refer to an "asymptotic expansion", we mean an "asymptotic expansion for large values".

Two cases arise.

(i) If P is at the top of a mountain and Q at the bottom of the same one (Fig. 15) then the maximum height is predetermined—and so you just slide down the mountain any way you wish. The reverse applies if the positions of P and Q are reversed—it's just harder work!

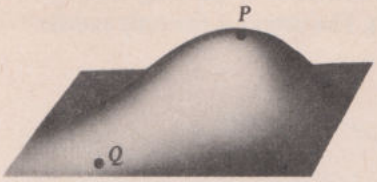


Fig. 15

(ii) But it may be the case that no such path exists—that whatever route you take from P to Q you have to do some climbing, every path from P to Q containing some points at which the height $|\phi(z)|$ is greater than its values at P and Q. The problem is—how do you choose the "best" path from these (Fig. 16)? Any climber will say that you have to cross a col (or a *saddle point* if he is a mathematical or horse-riding climber). And you cross the saddle point in such a way that you keep as low as possible all the time (Fig. 17).

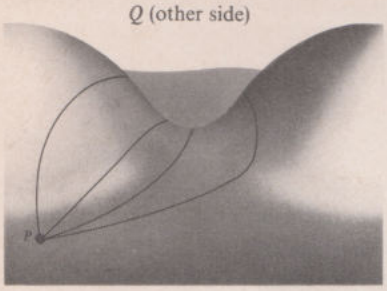


Fig. 16

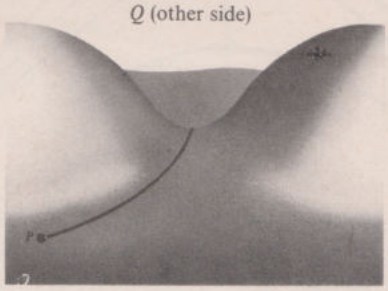


Fig. 17

Of course, there may be a choice of saddle points to cross between P and Q. In such a case you choose the lowest one. So the "mathematical" solution to our problem is to find all the saddle points on the surface—then choose the one at which $|\phi(z)|$ is smallest—and then choose a path which crosses this saddle point in such a way that it drops away from the saddle point as steeply as possible.

We shall consider only integrals of the form

$$\int_\Gamma e^{ph(z)} f(z)\, dz \tag{1}$$

because, as we shall see, they lead to integrals to which we can apply Watson's Lemma. (The steepest descent technique can be applied to other types of integral but it leads to real integrals which require techniques other than that given by Watson's Lemma for the calculation of their asymptotic expansions.)

For integrals of form (1) we look for the saddle points of the surface representing the function $z \longrightarrow |e^{ph(z)}|$ rather than the complete integrand. This is usually the term in the integrand that involves large changes when $|p|$ is large and therefore gives the contribution most sensitive to deformations of the contour.

Furthermore, if p is real and if $h(z) = u(x, y) + iv(x, y)$, then

$$|e^{ph(z)}| = e^{pu(x,y)},$$

and we can draw "contour lines" on which $|e^{ph(z)}|$ is constant by finding out where $u(x, y)$ is constant. (These contour lines are the level curves of the function $(x, y) \longrightarrow e^{pu(x,y)}$.)

Example 1

As an example, we shall take the function $h(z) = z^2$. We can get some idea of the geometry of the situation by writing

$$h(z) = u(x, y) + iv(x, y) = x^2 - y^2 + 2ixy.$$

On paths for which $x^2 - y^2$ is constant, $u(x, y)$ is constant. Regarding these as contour lines we get the following picture (Fig. 18). The origin is the only saddle point.

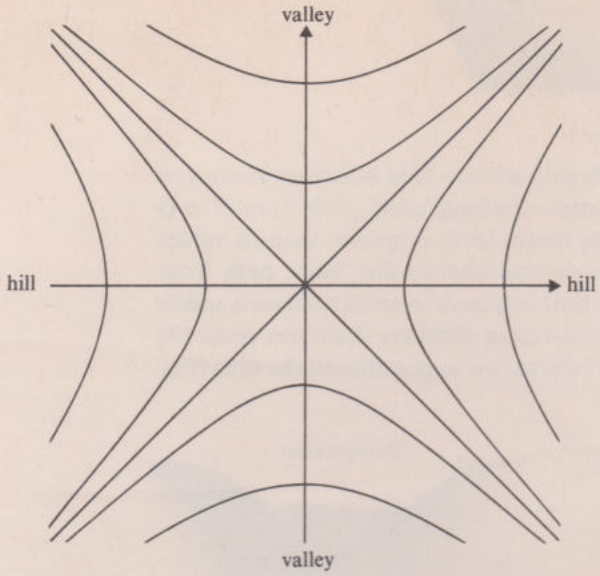


Fig. 18

To climb up or down any one or these hills as steeply as possible, we would try to cut across the contour lines at right angles. We now show mathematically that this is the case. The function $h(z) = z^2$ is analytic and one-one on some neighbourhood of every point except 0. Such a restriction of h has an analytic inverse whose derivative is never zero. From Section 3.2 of *Unit 3* we know that analytic functions preserve angles at points at which their derivatives are non-zero. In the uv-plane the coordinate lines u = constant, v = constant are orthogonal; therefore their images under the inverse function—namely the curves $u(x, y)$ = constant and $v(x, y)$ = constant—are orthogonal in the xy-plane. There are four paths starting from the origin (the saddle point) on which $v(x, y)$ is constant (Fig. 19); two of these are lines of steepest ascent (arrows pointing away from the origin) and the other two are lines of steepest descent.

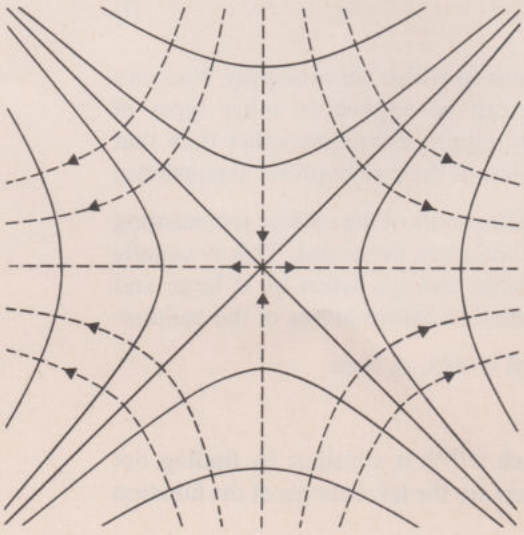


Fig. 19

Similar remarks to those in Example 1 apply in the general case where

$$h(z) = u(x, y) + iv(x, y).$$

If h is analytic and $h'(z_0) \neq 0$, then the curves $u(x, y) =$ constant and $v(x, y) =$ constant which pass through z_0 are orthogonal. (This fact may be proved by using the corollary to Theorem 8 of *Unit 16, Boundary Value Problems.*)

At a point where $h'(z) = 0$ we get a saddle point of the surface $w = u(x, y)$. (See Problem 2 of Section 14.10.) Suppose that at such a point $h(z) = a + ib$; then curves $v(x, y) = b$ leading from the saddle point are curves of steepest descent (or ascent) and along such curves $u(x, y)$ will decrease (or increase) steadily.

On a path L of steepest *descent*, $v(x, y) =$ constant, and $u(x, y)$ decreases steadily as we move away from the saddle point, and so we may write

$$u(x, y) = a - t,$$

where t increases from zero to some positive value T.

If we can deform the contour Γ to coincide with L without changing the value of the original integral (1), then

$$\int_\Gamma e^{ph(z)} f(z)\, dz = \int_L e^{ph(z)} f(z)\, dz,$$

and, since $u(x, y) = a - t$ on L and $v(x, y) = b$ on L, we have

$$\int_L e^{ph(z)} f(z)\, dz = \int_0^T e^{p(a-t+ib)} f(z) \frac{dz}{dt}\, dt, \text{ where } z \text{ is evaluated on } L.$$

In particular, if L extends to infinity (that is, L is a generalized contour), we get

$$\int_0^\infty e^{p(a-t+ib)} f(z) \frac{dz}{dt}\, dt = e^{p(a+ib)} \int_0^\infty e^{-pt} f(z) \frac{dz}{dt}\, dt.$$

This integral has the form of a Laplace transform and so we can apply Watson's Lemma if the following requirements hold:

$$\left| f(z) \frac{dz}{dt} \right| < Ce^{\alpha t}, t \geqslant 0, \text{ for some constants } \alpha \text{ and } C(>0)$$

and

$f(z)\dfrac{dz}{dt}$ has a Taylor series about 0 (in t) convergent in some interval.

This second requirement conceals a little more than might at first be thought. It would not normally be difficult to show that the Taylor series *exists*, for we know from *Unit 6* we need only look for singularities of $t \longrightarrow f(z)\dfrac{dz}{dt}$. But to *apply* Watson's Lemma we need to know what the Taylor series *is*, because we have to integrate it term by term with respect to t. You will see later (page 59) why this causes problems and how these can be overcome if we are willing to compromise with a poorer form of asymptotic formula (page 62).

Before we do that let us just try to be a little more convincing as to why a path through a saddle point should provide the answer to our problem of finding a path on which the maximum value of $|e^{ph(z)}|$ is a minimum.

First of all, since h is an analytic function, the surface representing $z \longrightarrow |e^{ph(z)}|$ has no maxima or minima. (This is a consequence of the Maximum Principle. See *Unit 9.*) Now consider any *closed* path on the surface. Again by the Maximum Principle at no point inside this path is there a point higher than the highest point of the path. On the other hand, a path through a saddle point must have

points on either side of it higher than the saddle point. We can deduce, therefore, that any *simple-closed* path through a saddle point must contain points higher than the saddle point.

We can now deduce that if a path Γ from A to B contains as its highest point a saddle point then any *other* path from A to B must contain a point at least as high as the saddle point. This follows by considering the *closed* path obtained by going from A to B along Γ and then back to A along the alternative route.

To develop this argument rigorously, we would have to take account of singularities and so on, but hopefully this argument is sufficient to convince you.

Example 2

Use the method of steepest descent to find an asymptotic expansion of $p \longrightarrow \int_\Gamma \frac{\exp(piz^2)}{(1-z)^2} dz$, where Γ is the "contour" $\operatorname{Re} z = 0$, $\operatorname{Im} z \geqslant 0$ (Fig. 20).

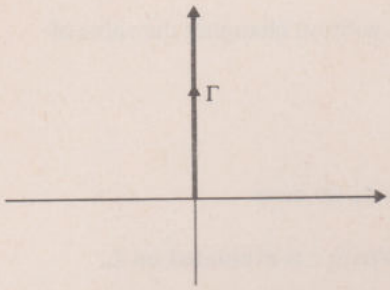


Fig. 20

Comparing this with the general case (see (1) on page 55), we have

$$h(z) = iz^2;$$

so

$$u(x, y) = -2xy, \quad v(x, y) = x^2 - y^2,$$

and

$$h'(z) = 2iz$$
$$= 0, \quad \text{if } z = 0.$$

Thus 0 is the saddle point and so the "contour" Γ already contains the saddle point. At the saddle point,

$$v(x, y) = x^2 - y^2 = 0$$

and so paths of steepest descent and ascent of $u(x, y)$ are given by

$$x^2 - y^2 = 0,$$

and so they are the lines

$$x + y = 0, \quad x - y = 0.$$

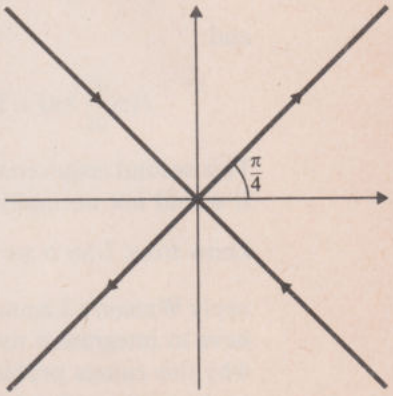


Fig. 21

On the line $x - y = 0$, we have $u(x, y) = -2xy = -2x^2$, and so $u(x, y)$ decreases as x increases and so this is the line of steepest *descent* (Fig. 21). We can deform Γ to coincide with the contour $L = \{(x, y)\colon x - y = 0,\ x > 0\}$, without crossing a singularity of the integrand, and so

$$\int_\Gamma \frac{\exp(piz^2)}{(1-z)^2} dz = \int_L \frac{\exp(piz^2)}{(1-z)^2} dz.$$

The contour L is parametrized by $h(z) = -t$ (because, on L, $u(x, y) = 0 - t$ and $v(x, y) = 0$) and we get

$$\int_L \frac{\exp(piz^2)}{(1-z)^2} dz = \int_0^\infty \frac{e^{-pt}}{(1-z)^2} \frac{dz}{dt} dt.$$

This is where we hit the snag referred to just before this example. To apply Watson's Lemma we have to express $\dfrac{1}{(1-z)^2}\dfrac{dz}{dt}$ as a Taylor series about 0 (in t).

In this (well-chosen) example we can obtain the Taylor series as follows.

We have

$$\frac{1}{(1-z)^2}\frac{dz}{dt} = \frac{d}{dt}\left(\frac{1}{1-z}\right)$$

and the integral becomes

$$\int_0^\infty e^{-pt}\frac{d}{dt}\left(\frac{1}{1-z}\right)dt.$$

But on L, $h(z) = -t$ and since $h(z) = iz^2$, we have

$$z^2 = it \text{ or } z = e^{i\pi/4}t^{1/2} \text{ on } L.$$

Thus, on L,

$$\frac{1}{(1-z)} = 1 + z + z^2 + z^3 + \cdots \quad (|z| < 1),$$

$$= 1 + (e^{i\pi/4}t^{1/2}) + (e^{i\pi/4}t^{1/2})^2 + (e^{i\pi/4}t^{1/2})^3 + \cdots \quad (|t| < 1),$$

and so

$$\frac{d}{dt}\left(\frac{1}{1-z}\right) = \frac{e^{i\pi/4}}{2t^{1/2}}\sum_{k=0}^{\infty}(k+1)(e^{i\pi/4}t^{1/2})^k$$

and this series also converges for $|t| < 1$.

Thus in

$$\int_0^\infty e^{-pt}\frac{d}{dt}\left(\frac{1}{1-z}\right)dt$$

we can use the Taylor series for $\dfrac{d}{dt}\left(\dfrac{1}{1-z}\right)$ and integrate term by term to obtain the asymptotic expansion. The kth term will be

$$\int_0^\infty e^{-pt}\frac{e^{i\pi/4}}{2t^{1/2}}(k+1)(e^{i\pi/4}t^{1/2})^k\,dt = \int_0^\infty \frac{(k+1)}{2}e^{i(k+1)\pi/4}e^{-pt}t^{(k-1)/2}\,dt.$$

The integral

$$\int_0^\infty e^{-pt}t^{(k-1)/2}\,dt$$

certainly exists for k an integer: the substitution $u = pt$ gives

$$\frac{1}{p^{(k+1)/2}}\int_0^\infty e^{-u}u^{(k-1)/2}\,du.$$

The integral is now of the form used to define the gamma function, and it reduces to

$$\frac{\Gamma\big((k+1)/2\big)}{p^{(k+1)/2}}.$$

We can now write down the asymptotic expansion we have been looking for: the kth term ($k = 0, 1, 2, \ldots$) is

$$\frac{(k+1)e^{i(k+1)\pi/4}}{2}\cdot\frac{\Gamma\big((k+1)/2\big)}{p^{(k+1)/2}} = \frac{e^{i(k+1)\pi/4}\Gamma\big((k+3)/2\big)}{p^{(k+1)/2}},$$

since $n\Gamma(n) = \Gamma(n+1)$.

Thus, starting the sum from 1 rather than 0, we have

$$\int_0^\infty \frac{\exp(piz^2)}{(1-z)^2}dz \sim \sum_{k=1}^{n} e^{ki\pi/4}p^{-k/2}\Gamma\big((k+1)/2\big).$$

(Note that the series $\sum_{k=1}^{\infty} e^{ki\pi/4}p^{-k/2}\Gamma\big((k+1)/2\big)$ diverges.)

Strictly speaking, in cases such as this, where the "contour" is infinite, we should check that the contour *can be* deformed, even if a singularity is not crossed. In our examples, we shall assume that we can do this.

Example 3

Find an asymptotic expansion for $\int_1^\infty \exp\big(ip(z^3-3z)\big)\,dz$.

Comparing this with the general case, we have

$$h(z) = i(z^3 - 3z)$$

and

$$h'(z) = 3i(z^2 - 1)$$
$$= 0, \quad \text{if } z = \pm 1.$$

We therefore have saddle points at ± 1. Since 1 is the starting point of the contour, we shall consider the saddle point at 1. At this point, $h(z) = -2i$. If we write

$$h(z) = u(x, y) + iv(x, y),$$

then, at the saddle point 1

$$u(x, y) = 0 \quad \text{and} \quad v(x, y) = -2.$$

Paths with equation $v(x, y) = -2$ will be paths of steepest descent or ascent for $u(x, y)$ from the saddle point. Since

$$h(z) = -(3x^2y - y^3 - 3y) + i(x^3 - 3xy^2 - 3x),$$

these paths are given by

$$x^3 - 3xy^2 - 3x = -2,$$

and are shown in Fig. 22.

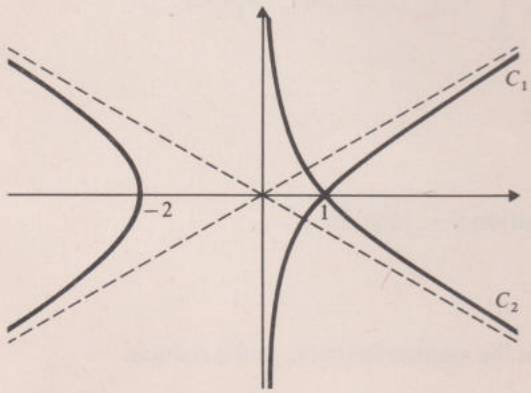


Fig. 22

(See the note after this example if you do not know why the graph looks like this.)

Let C_1 and C_2 be the two branches of the graph which pass through 1 (see Fig. 22). We have to decide which of the four paths starting from 1 are descent paths and which ascent paths.

We have

$$u(x, y) = -3x^2y + y^3 + 3y,$$

and on C_1 and C_2

$$x^3 - 3xy^2 - 3x = -2.$$

When x is large, for points on C_1 and C_2 we have

$$x^2 \simeq 3y^2.$$

Thus for large values of x on C_1 and C_2, we have

$$u(x, y) \simeq -8y^3 + 3y.$$

On the contour C_1, $u(x, y)$ becomes large and negative as x increases and on C_2, $u(x, y)$ becomes large and positive as x increases. So we select C_1 (with $x \geqslant 1$) as the line of steepest *descent*. (Notice that with $|y|$ increasing and x becoming small, we deduce from $u(x, y) = -3x^2y + y^3 + 3y$ that C_1 is again a path of steepest descent on this side of the saddle point and C_2 a path of steepest descent. In fact, it can be proved that lines $v(x, y) =$ constant from a saddle point of the surface $w = u(x, y)$ alternate as paths of steepest descent and ascent; in general, of course, there may be more than four.)

Since $f: z \longrightarrow \exp(ip(z^3 - 3z))$ is an entire function, the "contour" of integration $\{z: \operatorname{Re} z = 1, \operatorname{Im} z \geqslant 0\}$ can be deformed to coincide with the path of steepest descent, along C_1 with x increasing. Call this path Γ. Then

$$\int_1^\infty \exp(ip(z^3 - 3z))\, dz = \int_\Gamma \exp(ip(z^3 - 3z))\, dz,$$

and we proceed to try to find an asymptotic expansion for the right-hand integral by comparing it with our standard integral $\int_\Gamma e^{ph(z)} f(z)\, dz$; we have $f(z) = 1$ and $h(z) = i(z^3 - 3z)$.

We have seen that at the saddle point $z = 1$, we have $h(z) = -2i$ and so $u(x, y) = 0$ and $v(x, y) = -2$.

We have shown that on Γ, $v(x, y)$ remains constant and $u(x, y)$ decreases from 0. If we let

$$h(x, y) = -t - 2i,$$

then t increases from zero as we move along Γ and the integral becomes

$$e^{-2ip} \int_0^\infty e^{-pt} \frac{dz}{dt}\, dt.$$

This is of a form to which we can apply Watson's Lemma, provided we can express $\dfrac{dz}{dt}$ as a power series about 0 (in t).

When we arrived at this stage in the last example, we were able to perform some algebraic juggling to obtain such an expansion. This example is not so easy, so we shall return to it after we have developed a little more theory (page 64).

Note on the Shape of the Curves in Example 3

To find the general form of the paths defined by

$$x^3 - 3xy^2 - 3x = -2$$

we can proceed as follows.

(1) If (x, y) satisfies the equation so does $(x, -y)$, so the paths are symmetric about the x-axis.

(2) Paths cross the x-axis where

$$x^3 - 3x + 2 = 0$$

which gives $(x - 1)^2(x + 2) = 0$.

(3) As x approaches 0 from 1, y^2 becomes large.

(4) As x increases from 1, we can use the rearrangement

$$y^2 = \frac{x^3 - 3x + 2}{3x}$$

to deduce that the paths approach the lines $y = \pm \frac{x}{\sqrt{3}}$.

(5) When $-2 < x < 1$, $x^3 - 3x + 2$ is positive and so when $-2 < x < 0$, y is not defined. There are no points (x, y) on the paths for $-2 < x < 0$.

(6) As x gets large and negative the paths approach the lines

$$y = \pm \frac{x}{\sqrt{3}}.$$

The Dominant Term

Although the problem of inversion of a power series (that is, determining the power series of the inverse of a one-one function f, given the power series for f) has been solved (by Lagrange), the result is often technically difficult to apply. (See Problem 1 of Section 14.10.) It is fortunate, then, that it is very often good enough to obtain just the first term in an asymptotic expansion. The situations that may arise can be put roughly into three classes. The most general situation is when the saddle point is not on the contour of integration and the contour is deformed so as to pass through the saddle point (with appropriate allowances made if this entails crossing a singularity). When the saddle point already lies on the contour two possibilities arise. It may be at an endpoint of the contour, or, as in the examples we have considered, it may be at some other point. In each of these two cases, it is possible to derive a formula that gives this first, or dominant, term in the asymptotic expansion—or at least the dominant contribution of the saddle point.

We shall obtain a formula for a class of integrals of the form

$$\int_\Gamma e^{ph(z)} f(z)\, dz, \quad p > 0.$$

As usual, let $h(z) = u(x, y) + iv(x, y)$. Suppose z_0 is a saddle point of the surface $w = u(x, y)$. Suppose also that z_0 lies on Γ (in other words, any deformation of the contour that is required has been carried out).

Since $h'(z_0) = 0$, the Taylor series for h about z_0 takes the form

$$h(z) = h(z_0) + \frac{h''(z_0)}{2!}(z - z_0)^2 + \cdots.$$

If $h''(z_0) \neq 0$ then by Taylor's Theorem we can write

$$h(z) - h(z_0) = (z - z_0)^2 g(z),$$

where g is analytic on a neighbourhood of z_0 and $g(z_0) = h''(z_0)/2!$.

If Γ_1 is the part of Γ that lies in this neighbourhood (Fig. 23), then

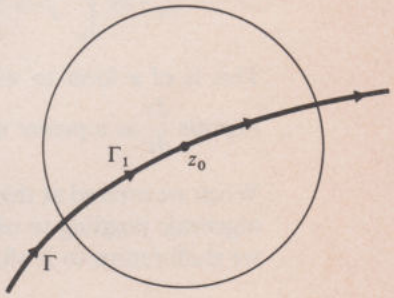


Fig. 23

$$\int_\Gamma e^{ph(z)} f(z)\, dz = \exp(ph(z_0)) \int_\Gamma \exp(p[h(z) - h(z_0)]) f(z)\, dz$$

$$\simeq \exp(ph(z_0)) \int_{\Gamma_1} \exp(ph''(z_0)(z - z_0)^2/2) f(z)\, dz.$$

We now try to put this integral into a form suitable for Watson's Lemma; in particular, we need a term e^{-pt} inside the integral, so we substitute

$$-t = \frac{h''(z_0)(z - z_0)^2}{2}.$$

This means that $-t$ is the first non-zero term in the Taylor series for $z \longrightarrow h(z) - h(z_0)$. On a path of steepest descent, the imaginary part of $h(z)$ is constant and so $h(z) - h(z_0)$ is real on such a path. This real quantity decreases continuously along the path (from z_0). The t we are using here is an approximation to the t that we substituted in the general statement of the method of steepest descent (page 57).

Taking a first approximation to t, means that we have to invert only the function

$$\phi: z \longrightarrow \frac{-h''(z_0)(z - z_0)^2}{2}$$

rather than a complete Taylor series.

The inverse of ϕ has two branches:

$$g_1(t) = z_0 + i\sqrt{t}\sqrt{\frac{2}{h''(z_0)}}, \quad t \in [0, \infty],$$

and

$$g_2(t) = z_0 - i\sqrt{t}\sqrt{\frac{2}{h''(z_0)}}, \quad t \in [0, \infty].$$

These correspond to the two paths of steepest descent on opposite sides of the saddle point. In Fig. 23, Γ_1 is the path formed from the two paths of steepest descent from z_0. (It may help to consider again Example 3 where the paths of steepest descent from the point 1 are

$$y = \sqrt{\frac{x^3 - 3x + 2}{3x}} \quad \text{for } x > 1$$

and

$$y = -\sqrt{\frac{x^3 - 3x + 2}{3x}} \quad \text{for } x < 1,$$

and so the two paths of steepest descent from 1 come from

$$z = x \pm i\sqrt{\frac{x^3 - 3x + 2}{3x}},$$

forming two continuous curves through 1.)

The derivatives of g_1 and g_2 are

$$g_1'(t) = \frac{i}{\sqrt{t2h''(2_0)}} \quad \text{and} \quad g_2'(t) = \frac{i}{\sqrt{t2h''(z_0)}}.$$

The difference in sign between these two derivatives accounts for the fact that one has to take account of contributions on both sides of the saddle points: as one approaches and then passes through the saddle point, t first decreases (to zero) and then increases (remember that the real part of $h(z)$ is approximately $-t$, not t). The contribution to the integral on each side of the saddle point is thus approximated by

$$\exp(ph(z_0))\int_{\Gamma_1} e^{-pt} f\left(z_0 \pm i\sqrt{t}\sqrt{\frac{2}{h''(z_0)}}\right) g'(t)\, dt$$

where g' stands for either g_1' or $-g_2'$. The *total* effect is therefore twice this value, and is given by

$$I = \exp(ph(z_0))i\sqrt{\frac{2}{h''(z_0)}}\int_0^T e^{-pt} f\left(z_0 \pm i\sqrt{t}\sqrt{\frac{2}{h''(z_0)}}\right)\frac{1}{\sqrt{t}}\, dt,$$

where T is the value of t that takes us to the edge of the approximating neighbourhood. Taking the first term in the Taylor series for f and applying Watson's Lemma we get

$$I \sim \exp(ph(z_0))\, f(z_0) i \sqrt{\frac{2}{h''(z_0)}} \int_0^T \frac{e^{-pt}}{\sqrt{t}}\, dt,$$

where we have extended the meaning of $\sim$ in the obvious way.

The next move is to relate $\displaystyle\int_0^T \frac{e^{-pt}}{\sqrt{t}}\, dt$ to an integral we know something about.

First of all we eliminate the p by substituting $u = pt$, to give

$$\frac{1}{\sqrt{p}} \int_0^U \frac{e^{-u}}{\sqrt{u}} du, \quad \text{where } U = pT.$$

Now

$$\int_0^\infty \frac{e^{-u}}{\sqrt{u}} du$$

is an integral whose value is $\sqrt{\pi}$. (See Self-Assessment Question 5 of Section 11.8 of *Unit 11*.) Furthermore,

$$\int_0^U \frac{e^{-u}}{\sqrt{u}} du = \int_0^\infty \frac{e^{-u}}{\sqrt{u}} du - \int_U^\infty \frac{e^{-u}}{\sqrt{u}} du,$$

and

$$\int_U^\infty \frac{e^{-u}}{\sqrt{u}} du \leqslant \frac{1}{\sqrt{U}} \int_U^\infty e^{-u}\, du = \frac{e^{-U}}{\sqrt{U}}.$$

Thus,

$$\int_0^T \frac{e^{-pt}}{\sqrt{t}} dt \simeq \frac{\sqrt{\pi}}{\sqrt{p}}$$

with an error less than $e^{-pt}/\sqrt{p\,T}$ and the approximation is "good" for large p. We deduce then that

$$\frac{i \exp(ph(z_0))\, f(z_0)}{\sqrt{p}} \cdot \sqrt{\frac{2\pi}{h''(z_0)}} \tag{2}$$

gives the **dominant term in the contribution from the saddle point z_0 to the asymptotic expansion** of the integral, when Γ can be deformed to pass through z_0. If the saddle point is an end point of Γ, then the contribution is half of that given above.

Example 4

The dominant term in the asymptotic expansion for the integral in Example 3 is (from (2) and employing the factor $\frac{1}{2}$)

$$\frac{1}{2} \cdot \frac{ie^{-2pi}}{\sqrt{p}} \cdot \sqrt{\frac{2\pi}{6i}} = \frac{1}{2}\sqrt{\frac{\pi}{3}}\, p^{-1/2} e^{i(\pi/4 - 2p)}.$$

Example 5

Consider the function $p \longrightarrow \Gamma(p+1)$, where

$$\Gamma(p+1) = \int_0^\infty e^{-u} u^p\, du, \quad p > -1.$$

We shall use formula (2) to obtain the dominant term of the asymptotic expansion to $\int_0^\infty e^{-u}u^p\,du$. We must first transform the integral into the required form:

$$\int_\Gamma e^{ph(z)}f(z)\,dz,$$

where Γ passes through the saddle points of the surface representing $\mathrm{Re}\,(h(z))$. To get this form, we write

$$u = pz.$$

This gives

$$\int_0^\infty e^{-pz}(pz)^p p\,dz = p^{p+1}\int_0^\infty e^{p(-z+\mathrm{Log}\,z)}\,dz.$$

This is the required form, with $f(z) = 1$ and

$$h(z) = -z + \mathrm{Log}\,z.$$

We have

$$h'(z) = -1 + \frac{1}{z},$$

and so there is just one saddle point, at $z = 1$, which lies on the path of integration.

Substituting in formula (2), we obtain

$$\Gamma(p+1) \sim p^{p+1}\,e^{-p}\sqrt{\frac{2\pi}{p}},$$

and if p is an integer

$$p! \sim p^{p+1}e^{-p}\sqrt{\frac{2\pi}{p}},$$

which is known as *Stirling's Formula.*

Summary

We have seen the method of steepest descent applied to the calculation of asymptotic expansions of the form $\int_\Gamma e^{ph(z)}f(z)\,dz$ and derived a formula for the first (or dominant) term in that expansion.

14.10 PROBLEMS

1. Let f be a one–one function and let F be the inverse of f. Show that if

$$f(z) = w = a_1 z + a_2 z^2 + \cdots$$

and

$$F(w) = z = b_1 w + b_2 w^2 + \cdots,$$

then

$$b_n = \frac{1}{n} \operatorname{Res}(z \longrightarrow 1/[f(z)]^n, 0).$$

(Hint: Use Taylor's Theorem to write b_n as an integral. Let $z = F(w)$; use this substitution in the integral for b_n and then integration by parts to obtain the result.)

2. Show that if the function $h(z) = u(x, y) + iv(x, y)$ is analytic, and u is not a constant function, then saddle points of the surface $w = u(x, y)$ occur at points for which $h'(z) = 0$.
(Hint: See Problem 3 of Section 9.2 of *Unit 9*.)

3. Find suitable paths of steepest descent (or ascent) for

$$\int_\Gamma e^{p(\sinh z - z)}\, dz,$$

where Γ is the "contour" in Fig. 24.

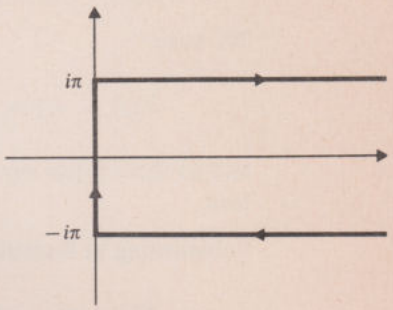


Fig. 24

4. For Example 3 of Section 14.9, on the path of steepest descent we have

$$u(z) = -t - 2i.$$

By substituting $Z = z - 1$ and then $t = -3iT^2$, show that

$$T = Z(1 + Z/3)^{1/2}.$$

Use the result of Problem 1 to show that if

$$Z = \sum_{n=1}^{\infty} b_n T^n$$

then

$$b_n = \frac{(-1)^{n-1} \dfrac{n}{2}\left(\dfrac{n}{2} + 1\right) \cdots \left(\dfrac{n}{2} + (n-2)\right)}{n \cdot (n-1)! 3^{n-1}}.$$

5. (a) Find a suitable path of steepest descent for

$$I = \int_\Gamma e^{ip \cosh z} \cosh nz\, dz,$$

where Γ is the path shown in Fig. 25 by going through the following steps.

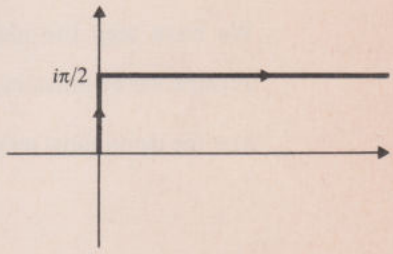


Fig. 25

(i) Show that 0 is a saddle point.

(ii) Show that the paths of steepest descent (or ascent) from 0 have equation $\cosh x \cos y = 1$.

(iii) Show that, on such paths, as y approaches $\pm\pi/2$, $|x|$ becomes large.

(iv) By using the first two terms of the Taylor series for cos about 0, show that for small x and y the curves are close to the lines $y = \pm x$.

(v) Sketch the paths of steepest descent (or ascent) and indicate a path of steepest descent.

(b) Show that on this path of steepest descent the integral becomes

$$\frac{e^{ip}}{n}\int_0^\infty e^{-pt}\cosh nz\,\frac{dz}{dt}\,dt$$

where $i\cosh z = i - t$.

6. (i) To obtain the complete asymptotic expansion for the integral I in Problem 5, we would have to express $\cosh nz\,\dfrac{dz}{dt}$ as a Taylor series in t.

The first term in the series may be obtained directly from formula (2) derived in Section 14.9.

If

$$Hk_n(p) = \frac{2e^{-n\pi i/2}}{\pi i}\,I,$$

show that

$$Hk_n(p) \sim \sqrt{\frac{2}{\pi p}}\cdot\exp\left(-i\left[p - \frac{n\pi}{2} - \frac{\pi}{4}\right]\right).$$

(ii) Defining $Hl_n(t)$ as the same integral as $Hk_n(t)$, but with Γ as in Fig. 26, show that

$$Hl_n(p) \sim \sqrt{\frac{2}{\pi p}}\cdot\exp\left(-i\left[p - \frac{n\pi}{2} - \frac{\pi}{4}\right]\right).$$

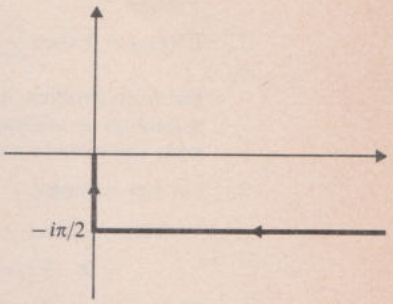


Fig. 26

7. (i) If $J_n(p) = \frac{1}{2}[Hk_n(p) + Hl_n(p)]$, where $Hk_n(p)$ and $Hl_n(p)$ are defined in Problem 6, draw the contour along which the integral for $J_n(p)$ is taken (corresponding to the contours for $Hk_n(p)$ and $Hl_n(p)$).

Show that

$$J_n(p) \sim \sqrt{\frac{2}{\pi p}}\cdot\cos\left(p - \frac{n\pi}{2} - \frac{\pi}{4}\right).$$

(ii) If $Y_n(p) = \dfrac{1}{2i}[Hk_n(p) - Hl_n(p)]$

show that

$$Y_n(p) \sim \sqrt{\frac{2}{\pi p}}\cdot\sin\left(p - \frac{n\pi}{2} - \frac{\pi}{4}\right).$$

(The functions J_n and Y_n are respectively called *Bessel functions* of the first and second kind of order n. The functions Hk_n and Hl_n are Bessel functions of the third kind, and are sometimes called *Hankel functions*. They arise in physics in many problems involving spherical symmetry. The asymptotic formulas for J_n and Y_n reveal some similarity with the trigonometric functions. This is enhanced by a comparison of the definitions of J_n and Y_n in terms of Hk_n and Hl_n with the formulas

$$\cos t = \frac{1}{2}(e^{it} + e^{-it})$$

$$\sin t = \frac{1}{2i}(e^{it} - e^{-it}).$$

Solutions

1. We have

$$b_n = \frac{F^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_C \frac{F(w)}{w^{n+1}}\,dw,$$

where C is a suitable circle.

Let $F(w) = z$ and note that $w = f(z)$; making this substitution in the integral, we obtain

$$b_n = \frac{1}{2\pi i}\int_C \frac{z\,f'(z)}{[f(z)]^{n+1}}\,dz$$

$$= \frac{z}{2\pi i n[f(z)]^n}\bigg|_C + \frac{1}{2\pi i n}\int_C \frac{dz}{[f(z)]^n}, \quad \text{by integration by parts.}$$

The only singularities of $z \longrightarrow 1/[f(z)]^n$ are poles at zero and so the first term is zero. The second term is simply $1/n$ Res$(z \longrightarrow 1/[f(z)]^n, 0)$ as required.

(See also Problem 5 of Section 13.2 of *Unit 13, Properties of Analytic Functions.*)

2. If $h'(z_0) = 0$, then $\dfrac{\partial u}{\partial x}(x_0, y_0) = 0$ and $\dfrac{\partial u}{\partial y}(x_0, y_0) = 0$.

But from Problem 3 of Section 9.2 of *Unit 9*, the function $u(x, y)$ cannot have a local maximum or minimum at (x_0, y_0) unless it is a constant function. The point (x_0, y_0) must therefore be a saddle point.

3. Let $h(z) = (\sinh z - z)$; then

$$h'(z) = (\cosh z - 1)$$

$$= 0, \quad \text{if } z = 0.$$

The point $z = 0$ is thus a saddle point.

Let $h(z) = u(x, y) + iv(x, y)$, then at $z = 0$, we have $v(x, y) = 0$. Thus paths with equation $v(x, y) = 0$ are paths of steepest descent, or ascent, from 0.

We have

$$u(x, y) + iv(x, y) = \sinh(x + iy) - (x + iy)$$

$$= (\sinh x \cos y - x) + i(\sin y \cosh x - y).$$

Thus the paths of steepest descent have equation

$$\sin y \cosh x - y = 0.$$

To see what they look like, note that

$$\cosh x = \frac{y}{\sin y},$$

and so as y approaches $\pm\pi$, x increases. Also, if (x, y) is on one of the curves, so are $(-x, y)$, $(-x, -y)$ and $(x, -y)$; so the curves are symmetric about the x-axis and the y-axis. For *small* x and y, we use approximations given by the Taylor series of $\cosh x$ and $\sin y$, to obtain

$$1 + \frac{x^2}{2} \simeq \frac{y}{y - \dfrac{y^3}{3!}},$$

and so

$$1 + \frac{x^2}{2} \simeq 1 + \frac{y^2}{6}$$

for x and y small. Thus, near the origin, the curves are something like those with equation

$$y = \pm\sqrt{3}x$$

The paths of steepest descent are sketched in Fig. 27.

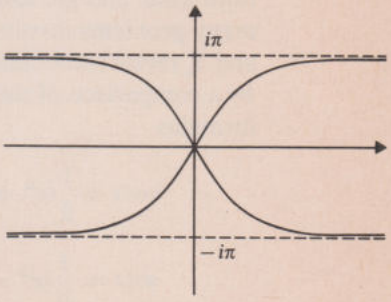


Fig. 27

4. At the saddle point $z = 1$, we have

$$u(x, y) = 0, \quad v(x, y) = -2.$$

Thus on the path of steepest descent,

$$h(z) = -t - 2i.$$

Thus

$$\begin{aligned} t &= -h(z) - 2i \\ &= -i(z-1)^2(z+2), \quad \text{since } h(z) = i(z^3 - 3z) \\ &= -iZ^2(Z+3), \quad \text{where } Z = z - 1. \end{aligned}$$

(The object of this manoeuvre is to make t and Z small together, so that the result of Problem 1 is applicable.)

$$t = -3iZ^2 - iZ^3.$$

If $t = -3iT^2$, then

$$-3iT^2 = -3iZ^2 - iZ^3,$$

and so

$$T^2 = Z^2 + \frac{Z^3}{3}$$

and

$$T = Z\left(1 + \frac{Z}{3}\right)^{1/2}.$$

This further simplifies the problem by making $T \simeq Z$ for Z small. If $f(Z) = Z\left(1 + \frac{Z}{3}\right)^{1/2}$, and $Z = \sum_{n=1}^{\infty} b_n T^n$, then by Problem 1, we have

$$b_n = \frac{1}{n}\text{Res}\left(Z \longrightarrow \frac{1}{[f(Z)]^n}, 0\right)$$

$$= \frac{1}{n}\text{Res}\left(Z \longrightarrow \frac{1}{Z^n}\left(1 - \frac{n}{2}\left(\frac{Z}{3}\right) + \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2!}\left(\frac{Z}{3}\right)^2 - \cdots, 0\right)\right).$$

Since b_n is $\frac{1}{n}$ times the coefficient of Z^{n-1} inside the bracket, we have

$$b_n = \frac{(-1)^{n-1}\frac{n}{2}\left(\frac{n}{2}+1\right)\cdots\left(\frac{n}{2}+(n-2)\right)}{n\cdot(n-1)!3^{n-1}}.$$

5. (a) (i) Let $h(z) = i\cosh z$; then $h'(z) = 0$ if

$$i\sinh z = 0.$$

Thus the origin is a saddle point, and it is on Γ.

(ii) Now, if $z = x + iy$, we have

$$h(z) = -\sinh x \sin y + i\cosh x \cos y;$$

so if $h(z) = u(x, y) + iv(x, y)$, we have

$$v(x, y) = \cosh x \cos y.$$

At the origin, $h(z) = i$ and so $v(x, y) = 1$. The paths of steepest descent, or ascent, are thus given by

$$\cosh x \cos y = 1.$$

(iii) To see what these paths look like, note that if (x, y) is on a path so are $(-x, y)$, $(-x, -y)$ and $(x, -y)$. As y approaches $\pm\pi/2$, $|\cosh x|$ must become large so as to keep the product $\cosh x \cos y$ constant. Thus $|x|$ becomes large.

(iv) Near the origin, we have

$$\left(1 + \frac{x^2}{2}\right) \simeq \frac{1}{1 - \frac{y^2}{2}}$$

and so

$$\left(1 + \frac{x^2}{2}\right) \simeq 1 + \frac{y^2}{2}.$$

Thus for small x and y, the paths are close to the lines

$$y = \pm x.$$

(v) The paths are shown in Fig. 28. On these paths,

$$u(x, y) = -\sinh x \sin y$$

and so $u(x, y)$ is decreasing for $0 \leqslant y < \frac{\pi}{2}$ and x is positive.

Thus the path in the first quadrant, with an arrow (Fig. 28), is a path of steepest descent.

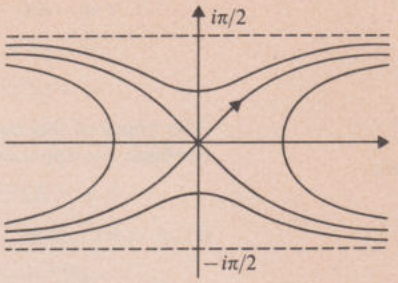


Fig. 28

(b) On paths of steepest descent (or ascent) $v(x, y) = 1$ and so $h(z) = u(x, y) + i$. Further, on a path of steepest descent, u decreases continuously from 0 and so, on such a path

$$h(z) = i - t$$

with t increasing from 0 as z moves away from the origin. The integral becomes

$$\frac{1}{n}\int_0^\infty e^{p(i-t)}\cosh nz \frac{dz}{dt}\,dt,$$

where $i\cosh z = i - t$, as required.

6. (i) Formula (2) of Section 14.9 relates to

$$\int_\Gamma e^{ph(z)} f(z)\,dz$$

and gives the dominant term in the asymptotic expansion, due to a saddle point at z_0, as

$$\frac{i\exp(ph(z_0))\,f(z_0)}{\sqrt{p}} \cdot \sqrt{\frac{2\pi}{h''(z_0)}};$$

in this case, $z_0 = 0$, $f(z) = \cosh nz$ and $h(z) = i\cosh z$, and we have to divide by 2 because the saddle point is an end point of the contour. We get

$$Hk_n(p) \sim \frac{1}{2} \cdot \frac{2e^{-n\pi i/2}}{\pi i} \cdot \frac{ie^{pi}}{\sqrt{p}} \sqrt{\frac{2\pi}{i}}.$$

Thus

$$Hk_n(p) \sim \sqrt{\frac{2}{\pi p}} \cdot \exp\left(i\left[p - \frac{n\pi}{2} - \frac{\pi}{4}\right]\right).$$

(This uses the fact that $\sqrt{1/i} = \pm e^{-n\pi i/4}$ and, as in the text derivation of the formula, we take the positive sign.)

(ii) Referring to Fig. 26, we see that it is easy to identify this path with a path of steepest *ascent* from the origin. On this path,

$$h(z) = i + t$$

with t increasing from 0. Instead of the integral in part (i), we get

$$\int_0^\infty e^{pt} \cosh nz \frac{dz}{dt}\,dt = \int_0^\infty e^{-(-p)t} \cosh nz \frac{dz}{dt}\,dt.$$

The total effect is simply to change the sign of p, giving the result.

7. (i) Fig. 29 gives the contour used to define $J_n(p)$.

$$J_n(p) \sim \frac{1}{2}\sqrt{\frac{2}{\pi p}} \left(\exp\left(i\left[p - \frac{n\pi}{2} - \frac{\pi}{4}\right]\right) + \exp\left(-i\left[p - \frac{n\pi}{2} - \frac{\pi}{4}\right]\right)\right)$$

$$= \sqrt{\frac{2}{\pi p}} \cdot \cos\left(p - \frac{n\pi}{2} - \frac{\pi}{4}\right).$$

(ii) As for (i), the result follows easily.

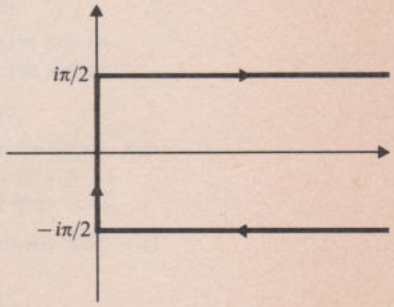


Fig. 29

APPENDIX: SOME RESULTS FROM REAL ANALYSIS

1. Definition of Uniform Convergence of an Integral

The integral $F(x) = \int_0^\infty f(x,y)\,dy$ **converges uniformly** (with respect to x) in the interval $[a, b]$ if, given any positive ε, there is a Y_0, dependent on ε but *not* on x such that, whenever $Y \geqslant Y_0$,

$$\left| \int_{Y_0}^\infty f(x, y)\,dy \right| < \varepsilon, \quad x \in [a, b].$$

(See also Example 2 of Section 11.3 of *Unit 11*.)

2. Test for Uniform Convergence of an Integral

The integral $\int_0^\infty f(x, y)\,dy$ converges uniformly on $[a, b]$ if an $M > 0$ and a $k > 1$ can be found such that for some Y_0,

$$|f(x, y)| < \frac{M}{y^k}, \quad y > Y_0, x \in [a, b].$$

3. Continuity

If the function f is continuous on the Cartesian product $[a, b] \times \{x : x \in \mathbf{R}, x > 0\}$ and $F(x) = \int_0^\infty f(x, y)\,dy$ is uniformly convergent on $[a, b]$, then F is continuous on $[a, b]$.

4. Integration with Respect to a Parameter

If $\int_0^\infty f(x, y)\,dy$ converges uniformly on $[a, b]$, then

$$\int_a^b \left(\int_0^\infty f(x, y)\,dy \right) dx = \int_0^\infty \left(\int_a^b f(x, y)\,dx \right) dy.$$

5. Differentiation with Respect to a Parameter

If the function $f(x, y)$ has a continuous partial derivative with respect to x on $[a, b]$ and if each of the integrals $\int_0^\infty \frac{\partial f}{\partial x}(x, y)\,dy$ and $F(x) = \int_0^\infty f(x, y)\,dy$ converge uniformly, then

$$F'(x) = \int_0^\infty \frac{\partial f}{\partial x}(x, y)\,dy.$$

(See also Example 2 of Section 11.3 of *Unit 11*.)

NOTATION

The following items of notation are explained on the pages given. Most other items of notation used in the text may be found in one of the following: *Mathematical Handbook* for M100, The Mathematic Foundation Course; *Handbook* for M231, Analysis; *Units 1, 2, 3, 4, 5, 6, 8, 9, 10 and 11* of M332, Complex Analysis.

F_s 9, 14

F_c 9, 13

$\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial x}(a, b)$ 14

F 19

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COMPLEX ANALYSIS

1 Complex Numbers

2 Continuous Functions

3 Differentiation

4 Integration

5 Cauchy's Theorem I

6 Taylor Series

7 NO TEXT

8 Singularities

9 Cauchy's Theorem II

10 The Calculus of Residues

11 Analytic Functions

12 NO TEXT

13 Properties of Analytic Functions

14 Laplace Transforms

15 Number Theory

16 Boundary Value Problems

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